MA 266 Final Exam

Fall 2008, version 2

Print YOUR LAST NAME: ____________________ FIRST NAME: ____________________

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Instructions:

1. This exam contains 21 pages, including the cover page and a table of Laplace transforms. The last two pages are left intentially blank, which you may use as scrap paper.
2. This exam consists of two parts: (a) 17 Multiple Choice Questions and (b) 7 Written Answer Questions:
   (a) Each of Problems # 1–17 contains a multiple choice question. Circle the answers to the Multiple Choice Questions and fill in the corresponding answers on your scantron. Each question worths 10 points.
   (b) Each of Problems # 11–17 contains an additional written answer question, labelled **Written Answer Question (a)–Written Answer Question (g)**. Show *detailed work* for these questions below the questions and circle your final answers. No credit will be given to no work. Your instructor will hand grade these questions and their weights were previously announced in class.

   Turn in *both* the scantron and the exam to your instructor in the end of the exam.
3. Put away books, notes, calculators, cell phones, i-pods and other electronic devices. No discussion during the exam.

1. Let y be the solution to the initial value problem:

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 0; \quad f(t) = \begin{cases} t, & 0 \le t < \pi \\ t + \sin(2t), & \pi \le t \end{cases}$$

Then the Laplace transform $\mathcal{L}\{y\}$ is?

A. $\dfrac{1}{s^2(s^2+1)} + e^{\pi s}\dfrac{2}{(s^2+4)(s^2+1)}$,

B. $\dfrac{1}{s^2(s^2+1)} + e^{-\pi s}\dfrac{1}{(s^2+4)(s^2+1)}$,

C. $\dfrac{1-e^{-\pi s}}{s^2(s^2+1)} + e^{-\pi s}\dfrac{2}{(s^2+4)(s^2+1)}$,

D. $\dfrac{1}{s^2(s^2+1)} + e^{-\pi s}\dfrac{2}{(s^2+4)(s^2+1)}$,

E. None of the above.

2. Which of the following is an implicit solution to the initial value problem

$$x^2 - 3y^2 - \sin(x+y) - \big(6xy + \sin(x+y)\big)y' = 0, \quad y(-1) = 1?$$

A. $x^3/3 - 3xy^2 + \cos(x+y) = 11/3$,

B. $x^2y + \cos(x+y) - y^3 = 1$,

C. $xy^2 + \cos(x+y) = 0$,

D. $x^2y - \cos(x+y) + 2 = 0$,

E. None of the above.

3. Find the Laplace transform of the function

$$f(t) = \int_0^t \sin(t-\tau)e^{\tau}\cos(\tau)\,d\tau.$$

A. $\dfrac{s}{(s^2+1)((s-1)^2+1)}$,

B. $\dfrac{s-1}{(s^2+1)((s-1)^2+1)}$,

C. $\dfrac{s}{((s-1)^2+1)^2}$,

D. $\dfrac{s-1}{((s-1)^2+1)^2}$,

E. $\dfrac{s}{(s^2+1)^2}$.

4. How many of the following differential equations are linear?

(1) $ty' + y - t^2 = \sin t$.

(2) $ty' + t - y^2 = \sin t$.

(3) $ty' + y = \sin y$.

(4) $ty'' + (\sin t)y = e^y$.

(5) $yy'' = \cos t$.

(6) $y' = y^{-1}$.

Answer:

A. None,

B. 1,

C. 2,

D. 3,

E. more than 3.

5. Find the asymptotically stable equilibrium solution(s) for $y' = y(y-1)^2(y-2)$.

A. $y(t) = 0$,

B. $y(t) = 1$,

C. $y(t) = 0, 1$,

D. $y(t) = 1, 2$,

E. $y(t) = 0, 2$.

6. The largest interval in which the solution of the initial value problem

$$(\cos t)y'' + t^2 y' - \frac{5}{t}y = \frac{e^t}{t-3}, \quad y(4) = 2, \quad y'(4) = 0,$$

is guaranteed to exist by the Existence and Uniqueness Theorem is:

A. $(0, \infty)$

B. $(\frac{\pi}{2}, 3)$

C. $(0, \frac{\pi}{2})$

D. $(3, \frac{3\pi}{2})$

E. None of the above.

7. Find the general solution of the system

$$\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

A. $c_1 \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} e^{-t}$,

B. $c_1 \begin{pmatrix} -2\sin(2t) \\ \cos(2t) \end{pmatrix} e^{t} + c_2 \begin{pmatrix} 2\cos(2t) \\ \sin(2t) \end{pmatrix} e^{t}$,

C. $c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} \right]$,

D. $c_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{t} + c_2 \left[\begin{pmatrix} -2 \\ 1 \end{pmatrix} te^{t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{t} \right]$,

E. None of the above.

8. Find the general solution to $y'' + 2y' = 3 + 2\sin(2t)$.

A. $y = c_1 + c_2 e^{-2t} + \frac{3}{2}t - \frac{1}{2}\sin(2t)$

B. $y = c_1 \sin(2t) + c_2 \cos(2t) + \frac{3}{2} - \frac{1}{4}t\,(\sin(2t) + \cos(2t))$

C. $y = c_1 e^{2t} + c_2 + \frac{3}{2}t - \frac{1}{4}\sin(2t) + \frac{1}{4}\cos(2t)$

D. $y = c_1 + c_2\, e^{-2t} + \frac{3}{2}t - \frac{1}{4}\sin(2t) - \frac{1}{4}\cos(2t)$

E. None of the above.

9. According to the method of undetermined coefficients, what is the proper form of a particular solution Y to the following differential equation?

$$y''' + 5y'' + 3y' - 9y = 5e^{-3t} - 9t.$$

A. $Y(t) = Ae^{-3t} + Bt$

B. $Y(t) = Ate^{-3t} + Bt + C,$

C. $Y(t) = At^2e^{-3t} + Bt + C,$

D. $Y(t) = Ae^t + Be^{-3t} + Cte^{-3t} + Dt.$

E. None of the above.

Hint: $r^3 + 5r^2 + 3r - 9 = (r-1)(r+3)^2$.

10. A mass weighing 4 lb, when hung from the ceiling, streches a spring $\frac{1}{2}$ ft at equilibrium. Suppose that the mass is pulled down an additional $\frac{1}{4}$ ft from the equilibrium point, and then set in motion with a upward velocity of 8 ft/sec. (The gravitational constant $g = 32$ ft/sec^2.) If there is a damping force with a damping constant $\gamma = \frac{1}{2}$ lb-sec/ft and no external force. Suppose that t seconds later, the mass is displaced from its equilibrium position by $u(t)$ ft, taking the downward direction as the positive direction. Then $u(t)$ satisfies the initial value problem:

A. $4u'' + \dfrac{1}{2}u' + 8\,u = 0;\ u(0) = \dfrac{1}{4},\ u'(0) = 8$

B. $u'' + 4u' + 64\,u = 0;\ u(0) = \dfrac{1}{4},\ u'(0) = -8$

C. $\dfrac{1}{8}u'' + \dfrac{1}{2}u' + 8\,u = 0;\ u(0) = -\dfrac{1}{4},\ u'(0) = -8$

D. $4u'' + \dfrac{1}{2}u' + 8\,u = 0;\ u(0) = \dfrac{1}{4},\ u'(0) = 0$

E. $\dfrac{1}{8}u'' + 8\,u = 0;\ u(0) = -\dfrac{1}{4},\ u'(0) = -8.$

11. Which of the following functions below is a particular solution to the differential equation

$$y'' + y = \frac{1}{\cos t}?$$

A. $y(t) = \frac{1}{2}e^t \int \frac{e^{-t}}{\cos t} dt - \frac{1}{2}e^{-t} \int \frac{e^t}{\cos t} dt$

B. $y(t) = \cos^2 t + t \sin t$

C. $y(t) = \cos t \, \ln|\cos t| + t \sin t$

D. $y(t) = \ln|\cos t| + t \sin t$

E. None of the above.

Written Answer Question (a): Find the *general* solution to the above differential equation using the method of variation of parameters.

Hint: $$\int (\cos t)^n \sin t \, dt = \begin{cases} -\dfrac{(\cos t)^{n+1}}{n+1} + C & \text{when } n \neq -1 \\ -\ln|\cos t| + C & \text{when } n = -1 \end{cases}.$$

12. Let $y(x)$ be the solution to the initial value problem::

$$y' = \frac{y}{x} - \frac{x}{2y}, \; x > 0, \quad y(1) = 1.$$

Then $y(e^{-1}) =$?

A. 2,

B. $\sqrt{2}$,

C. e^{-1},

D. $\sqrt{2}/e$,

E. None of the above.

Hint: Notice that the right hand side of the differential equation only depends on the ratio y/x.

Written Answer Question (b): Write down your solution for $y(t)$. (You may leave it in implicit form).

13. Find the solution of the initial value problem

$$y'' - 3y' + 2y = \delta(t-2); \quad y(0) = 0, \quad y'(0) = 1.$$

A. $-e^{2t} + e^t + e^t u_2(t)(e^{2t-4} + e^{t-2})$,

B. $e^{2t} + e^t + u_2(t)(e^{2t-4} - e^{t-2})$,

C. $e^{2t} - e^t + u_2(t)(e^{2t-4} + e^{t-2})$,

D. $e^{2t} - e^t + u_2(t)(e^{2t-4} - e^{t-2})$,

E. $-e^{2t} + e^t + u_2(t)e^{t-2}$.

Written Answer Question (c): What is $\mathcal{L}\{y\}$?

14. A tank of capacity 30 liter (ℓ) initially contains 1 g of salt, which is dissolved in 20 ℓ of water. Brine containing 0.5 g/ℓ of salt flows into the tank at the rate of 2 ℓ/min, and the well-stirred mixture flows out of the tank at the rate of 1 ℓ/min. What is the concentration of the salt solution in the tank at the point of overflow? Express your answer in the unit g/ℓ.

A. $\frac{3}{4}(2 - 19\,e^{-1/2})$,

B. 0.3,

C. 9,

D. $1 - 9.5\,e^{-1/2}$,

E. None of the above.

Written Answer Question (d): What is the time of overflow? Write down a differential equation and initial conditions describing the amount of salt in the tank.

15. Which of the following is a particular solution to the system $\mathbf{x}' = \mathbf{A}\mathbf{x} + e^{-2t}\mathbf{g}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \qquad \mathbf{g} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

A. $(e^t - 2e^{-2t}/3)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1/9 \\ 1/3 \end{pmatrix}$,

B. $(e^t - e^{-2t})\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t}$,

C. $\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{-t} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}/2$,

D. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{t} - \begin{pmatrix} 1/3 \\ 1 \end{pmatrix} e^{-2t}$,

E. None of the above.

Written Answer Question (e): Write down the *general* solution to the above system.

Hint: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ are eigenvectors of $\mathbf{A}$ with eigenvalues 1 and -1 respectively.

16. If $y_1(t) = t^2$ is a solution of $t^2y'' - 3ty' + 4y = 0$ $(t > 0)$, use reduction of order to find a second solution $y_2(t)$.

A. $y_2(t) = t$

B. $y_2(t) = -\ln(t^3)$

C. $y_2(t) = -t^3$

D. $y_2(t) = t^2 \ln t$

E. None of the above.

Written Answer Question (f): Show detailed solution for the above problem using the reduction of order method.

17. In the phase portrait of the system

$$\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} \mathbf{x},$$

the origin is a:

A. improper node.

B. proper, asymptotically stable node.

C. proper, asymptotically unstable node.

D. spirial point.

E. saddle point.

Written Answer Question (g): Find the general solution to the above system.

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