Summer MA 15200 Lesson 21 Section 4.2 and Applied Problems

Remember the following information about inverse functions.

1. In order for a function to have an inverse, it must be one-to-one and pass a horizontal line test.
2. The inverse function can be found by interchanging *x* and *y* in the function's equation and solving for *y*.
3. If $f(a) = b,$ then $f^{-1}(b) = a$. The domain of *f* is the range of f^{-1} and the range of *f* is the domain of f^{-1}.
4. The compositions $f(f^{-1}(x))$ and $f^{-1}(f(x))$ both equal *x*.
5. The graph of f^{-1} is the reflection of the graph of *f* about the line $y = x$.

Because an exponential function is 1-1 and passes the horizontal line test, it has an inverse. This inverse is called a logarithmic function.

I Logarithmic Functions

According to point 2 above, we interchange the *x* and *y* and solve for *y* to find the equation of an inverse function.

$f(x) = b^x$ exponential function

$x = b^y$ inverse function How do we solve for *y*? There is no way to do this. Therefore a new notation needs to be used to represent an inverse of an exponential function, the logarithmic function.

> Definition of Logarithmic Function
> For $x > 0$ and $b > 0$ $(b \neq 1)$
>
> $y = \log_b x$ is equivalent to $x = b^y$
>
> The function $f(x) = \log_b x$ is the **logarithmic function with base *b*.**

The equation $y = \log_b x$ is called the logarithmic form and the equation $x = b^y$ is called the exponential form. The value of *y* in either form is called a **logarithm.** **<u>Note: The logarithm is an exponent.</u>**

Exponential Form

$x = b^y$

argument base exponent

Logarithmic Form

$y = \log_b x$

exponent base argument

> In this form, the *y* value representing the exponent is called a logarithm.

Ex 1: Convert each exponential form to logarithmic form and each logarithmic form to exponential form.

$a)\quad 3^4 = 81$

$b)\quad \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$c)\quad 25^{\frac{1}{2}} = 5$

$d)\quad 8^{-2} = \frac{1}{64}$

$h)\quad m^{rp} = x + 4$

$i)\quad (2a)^{y+7} = p^2$

$e)\quad \log_2 32 = 5$

$f)\quad \log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$

$g)\quad \log_5 \sqrt{5} = \frac{1}{2}$

$j)\quad \log_q(2mn) = 12$

$k)\quad \log_{x+3} 200 = rs$

II Finding logarithms

Remember: A logarithm is an exponent.

Ex 2: Find each logarithm.

$a)\quad \log_{10} 100,000$

$b)\quad \log_3 27$

$c)\quad \log_{20} 1$

$d)\quad \log_{15} 15$

$e)\quad \log_{12} \frac{1}{144}$

$f)$ $\log_4 64$

$g)$ $\log_{\frac{1}{2}} 32$

$h)$ $\log_3 81$

III Basic Logarithmic Properties

1. $\log_b b = 1$ Since the first power of any base equals that base, this is reasonable.
2. $\log_b 1 = 0$ Since any base to the zero power is 1, this is reasonable.

The exponential function $f(x) = b^x$ or $y = b^x$ and the logarithmic function $f^{-1}(x) = \log_b x$ or $y = \log_b x$ are inverses.
$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

This leads to 2 more basic logarithmic properties.

3. $\log_b b^x = x$ This is a composition function where $f(x) = b^x$ and $f^{-1}(x) = \log_b x$. $f^{-1}(f(x)) = f^{-1}(b^x) = \log_b b^x = x$ (the exponent)
4. $b^{\log_b x} = x$ This is a composition function where $f(x) = b^x$ and $f^{-1}(x) = \log_b x$. $f(f^{-1}(x)) = f(\log_b x) = b^{\log_b x} = x$ (the number or argument)

Ex 3: Simplify using the basic properties of logarithms.

$a)$ $\log_4 1 =$

$b)$ $\log_3 3 =$

$c)$ $12^{\log_{12} 4} =$

$d)$ $\log_{10} 10^5 =$

Ex 4: Simplify, if possible.

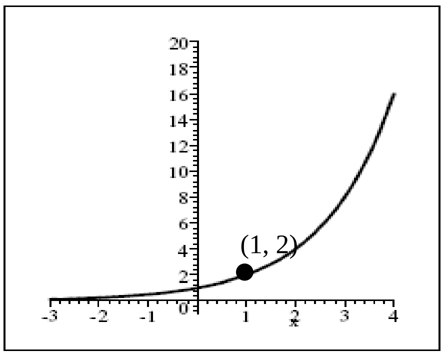
a) $\log_{(-4)} 1 =$

b) $\log(-100) =$

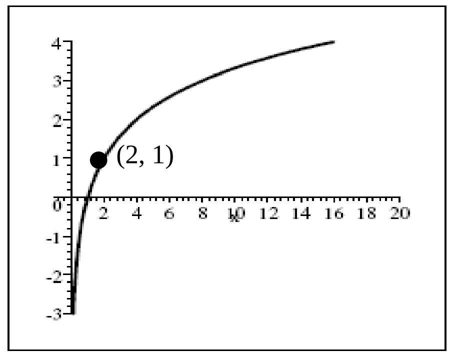
> Remember that bases must be positive and the argument values (the numbers) must be positive.

IV Graphs of Logarithmic Functions

Below is a graph of $y = 2^x$ and it inverse, $x = 2^y$.

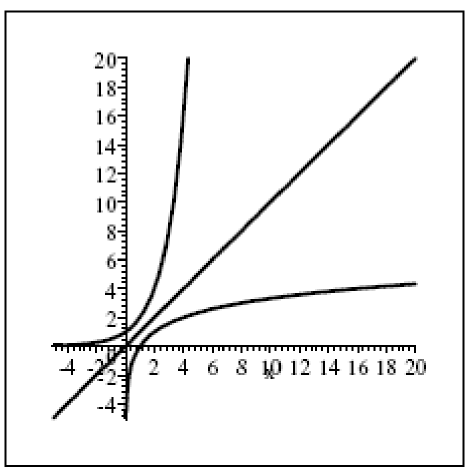


$y = 2^x$



$x = 2^y$

If you imagine the line $y = x$, you can see the symmetry about that line.
Below are both graphs on the same coordinate system along with $y = x$.



Characteristics of a logarithmic Graph:

The **inverse of this function**, $x = b^y$, has a graph with the following characteristics.

1. The ***x*-intercept** is (1, 0).
2. The graph still is increasing if $b > 1$, decreasing if $0 < b < 1$.
3. The **domain** is all positive numbers, so the graph is to the right of the *y*-axis. (The range is all real numbers.)
4. The ***y*-axis is an asymptote.**

V Common Logarithms

A logarithmic function with base 10 is called the **common logarithmic function.** Such a function is usually written without the 10 as the base.

$\log x$ is equivalent to $\log_{10} x$

A calculator with a key [log] can approximate common logarithms.

Put the number (argument) in the calculator, press the common log key.

<u>Ex 5:</u> Find each common logarithm **without a calculator**.

a) $\log 1000 =$

b) $\log \frac{1}{100} =$

c) $\log 0.001 =$

<u>Ex 6:</u> Use a calculator to approximate each common logarithm. Round to 4 decimal places.

a) $\log 0.025$

b) $\log 43.8$

Using the basic properties with base 10, we get the following properties.

1. $\log 10 = 1$
2. $\log 1 = 0$
3. $\log 10^x = x$
4. $10^{\log x} = x$

VI Natural Logarithms

A logarithm function with base *e* is called the **natural logarithmic function.** Such a function is usually written using ln rather than log and no base shown.

$\ln x$ is equivalent to $\log_e x$

A calculator with a key [ln] can approximate natural logarithms.

Put the number (argument) in the calculator, press the natural log key.

Ex 7: Use a calculator to approximate each natural logarithm. Round to 4 decimal places.

a) $\ln 0.988$

b) $\ln 2008$

Using the basic properties with base *e*, we get the following properties.

1. $\ln e = 1$
2. $\ln 1 = 0$
3. $\ln e^x = x$
4. $e^{\ln x} = x$

VII Modeling with logarithmic functions

The function $f(x) = 29 + 48.8\log(x+1)$ gives the percentage of adult height attained by a boy who is *x* years old.

Ex 8: Approximately what percentage of his adult height has a boy of age 11 acheived? (Notice: This model uses a common log.) Round to the nearest tenth of a percent.

The function $f(x) = 13.4\ln x - 11.6$ models the temperature increase in degrees Fahrenheit after *x* minutes in an enclosed vehicle when the outside temperature is from 72° to 96°.

Ex 9: Use the function above to approximate the temperature increase after 45 minutes. Round to the nearest tenth of a degree.

VIII Applied Problems

For part of your homework on this lesson, you will have a worksheet of applied problems that use exponential or logarithmic formulas. You will need to print off the worksheet, the formula sheet, and/or the answer sheet from the course web page (under other information, worksheets).

In order to solve the applied problems in this lesson, a student must know how to use the log, ln, e^x**,** and power key functions on a scientific calculator.

<u>Ex 10:</u> **Half-Life of an Element**
The **half-life** of an element is the amount of time necessary for the element to decay to half the original amount. Uranium is an example of an element that has a half-life. The half-life of radium is approximately 1600 years. The formula used to find the amount of radioactive material present at time *t*, where A_0 is the initial amount present (at $t = 0$), and h = half-life of the element is $A = A_0 2^{-\frac{t}{h}}$ *or* $A = A_0 2^{-t/h}$.

Tritium, a radioactive isotope of hydrogen, has a half-life of 12.4 years. If an initial sample has 50 grams, how much will remain after 100 years? Round to 4 decimal places.

A subscript of 0, such as A_0, means the amount initially or amount at time 0.

<u>Ex 11:</u> **Population Growth**
Another formula represents the population growth of lots of cities, towns, or countries. The formula is $P = P_0 e^{kt}$ where P is the final population, P_0 is the initial population at time 0, t is time in years, and $k = b - d$ (b is birth rate and d is death rate). k is a percent converted to a decimal. Note: In order for a country to grow k must be positive.

The population of a city is 45,000 in 2008. The birth rate is 11 per 1000 and the death rate is 8 per 1000, so the value of k is 0.011 - 0.008 = 0.003. What will be the population in 2018? Round to the nearest whole number.

Ex 12: **Light Intensity**

The intensity of light I (in lumens) at a distance of x meters below the surface of water is represented by $I = I_0 k^x$, where I_0 is the intensity of light above the water and k is a constant that depends on the clarity of the water.

At the center of a certain lake, the intensity of light above the water is 10 lumens and the value of k is 0.8. Find the intensity of light at 3 meters below the surface.

Ex 13: **Population Growth**

The population of Eagle River is growing exponentially according to the model, $P = 375(1.3)^t$, where t is years from the present date. Find the population in 6 years.

Ex 14: **Percent of Alcohol in Bloodstream**

For one individual the percent of alcohol absorbed into the bloodstream after drinking two shots of whiskey is given by $P = 0.3(1 - e^{-0.05t})$, where t is in minutes. Find the percent of alcohol in the bloodstream after ½ hour.

Ex 15: **Decibel Voltage Gain**
The measure of voltage gain of devices such as amplifiers or the length of a transmission line is called a decibel. If E_O is the output voltage of a device and E_i is the input voltage, the decibel voltage gain is given by db gain $= 20\log\frac{E_O}{E_i}$.

Find the db gain of an amplifier whose input voltage is 0.71 volt and whose output voltage is 20 volts.

Ex 16: **Earthquake Richter Scale**
The measurement of the intensity of an earthquake is a number from the Richter Scale. If *R* represents the intensity, *A* is the amplitude (measured in micrometers), and *P* is the period (time of one oscillation of the Earth's surface, measured in seconds), then $R = \log\frac{A}{P}$. (Notice the Richter scale is based on common logarithms. An earthquake of 4.0 is 10 times greater than one of 3.0.)

Find the intensity of an earthquake with an amplitude of 6000 micrometers and a period of 0.08 second. Round to the nearest tenth.

Ex 17: **Charging a Battery**
The time in minutes required to charge a battery depends on how close it is to being fully charged. If *M* is the theoretical maximum charge, *k* is a positive constant that is dependent upon the type of battery and charger, *C* is the given level of *M* to which the battery is being charged, then the time *t* required to reach that level is given by $t = -\frac{1}{k}\ln(1-\frac{C}{M})$. Note: *C* is usually a percentage of *M*. In the question below, $C = 0.4M$.

If $k = 0.201$, how long will it take a battery to reach a 40% charge? Assume that the battery was fully discharged when it began charging.

Ex 18: **Doubling Population**

If a population is growing exponentially at a certain annual rate, then the time required for that population to double is called the doubling time and is given by $t = \frac{\ln 2}{r}$, where t is time in years.

A town's population is growing at 9.2% per year. If this growth rate remains constant, how long will it take for the town's population to double?

Ex 19: **Annual Growth Rate**

If an investment is growing continuously for t years, its annual growth rate r is given by the formula $r = \frac{1}{t}\ln\frac{P}{P_0}$, where P is the current value and P_0 is the initial value of the investment.

An investment of $10,400 in America Online in 1992 was worth $10,400,000 in 1999. Find AOL's average annual growth rate during this period. Round to the nearest hundredth of a percent.