1 Section 10.4

1. Fill in the following two tables for addition and multiplication in 5-hour clock arithmetic.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	1	2
4	0	4	3	2	3

To obtain these values, we perform addition or subtraction in the usual way, and then divide the result by the number of hours on the clock (in this case 5: $0, 1, 2, 3$, and 4.) The *remainder* we get when performing this division is then our final answer. For example, $3 + 4 = 7$, and then 5 goes into 7 once with a remainder of 2. So in 5-hour clock arithmetic, $3+4 = 2$. Another example: $1+2 = 3$, and 5 goes into 3 zero times with a remainder of 3, so the answer is 3. (If the result of your initial addition is less than the number of hours on the clock, then your final clock-arithmetic result will coincide with the result you'd get from ordinary addition.)
Multiplication works in exactly the same way. We have $3 \times 3 = 9$, and 5 goes into 9 once with a remainder of 4, so $3 \times 3 = 4$ in 5-hour clock arithmetic.

2. Illustrate, with examples, that multiplication in the 5-hour clock system is commutative and associative.
Commutativity: I'm going to use the symbol $\otimes$ to denote 5-hour clock multiplication.
To find $3 \otimes 4$, we note that $3 \times 4 = 12$, and 5 goes into 12 twice with a remainder of 2. So $3 \otimes 4 = 2$.
To find $4 \otimes 3$, we note that $4 \times 3 = 12$, and then the remainder when dividing by 5 is again 2, so we conclude that $4 \otimes 3 = 2$.
Thus $4 \otimes 3 = 3 \otimes 4$.
(You could also just look at the multiplication table and observe that $a \otimes b = b \otimes a$ for every a, b under consideration.)

Associativity: We have $(3 \otimes 2) \otimes 4 = (1) \otimes 4 = 4$, while $3 \otimes (2 \otimes 4) =$

$3 \otimes (3) = 4$. Thus $(3 \otimes 2) \otimes 4 = 3 \otimes (2 \otimes 4)$.

3. Is the set closed under multiplication?
   Yes. All the numbers in the interior of the multiplication table are in the set $S = \{0, 1, 2, 3, 4\}$, which shows that multiplying two elements in S with five-hour clock multiplication will always yield an element of S.

4. Is there a multiplicative identity?
   Yes: 1.

5. Does every number in the system have a multiplicative inverse?
   Strictly speaking, no: 0 does not. But every nonzero element *does* have a multiplicative inverse. Recall that the multiplicative inverse of a number a is any b such that $b \times a = 1$. The element 0 has no multiplicative inverse (since the 0 row in the multiplication table contains no 1); but every other element does (since every other row has a 1 in it.) For example, the multiplicative inverse of 3 is 2, and vice-versa.

6. Finally, is multiplication distributive under addition? If so, this is a field.
   Let's see. Letting $\oplus$ denote 5-hour clock addition, distributivity means that we must have $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$ for every a, b, c on the clock. So let's try an example. We have
   $2 \otimes (3 \oplus 4) = 2 \otimes (2) = 4$, while
   $(2 \otimes 3) \oplus (2 \otimes 4) = 1 \oplus 3 = 4$. So
   $2 \otimes (3 \oplus 4) = (2 \otimes 3) \oplus (2 \otimes 4)$.
   We only tested one example, but distributivity does, in fact, hold. Therefore 5-hour clock arithmetic is a field. (As a little side-note, m-hour clock arithmetic is a field if and only if $m = p^n$ for some prime p. In this case $m = 5^1$, and 5 is prime so it's a field.)

7. We are also supposed to verify properties of addition.
   Additive identity: yes, 0.
   Additive closure: yes, all the elements in the interior of the addition table are indeed in the set $\{0, 1, 2, 3, 4\}$.
   Additive inverses: yes, there is a 0 in every row, so given any a in the set, we can find b with $a \oplus b = 0$.
   Associativity: trival to verify, but yes.
   Commutativity: ditto. For example, $3 \oplus 4 = 2 = 4 \oplus 3$.

2 Section 11.1

2. Is it possible for some composite numbers to also be prime numbers?
No. The definition of a composite number is "a positive integer which is neither 1 nor a prime." So by definition, a composite cannot be a prime.

3. An array for a prime number can have exactly one row or exactly one column. True or false?
True. If we have n rows, m columns and p total elements in the rectangular array, then $m \times n = p$. Then either m or n must be 1, because otherwise we'd be writing p as a product of two non-1 factors, and p wouldn't be prime.

4. Except for 1, only composite numbers are crossed out in the sieve of Eratosthenes. True or false?
True. See page 204 for a description of the sieve. The whole point of this sieve is that it crosses out composites and leaves the primes, so only the composites will be crossed out.

9. For each whole number variable below, write a sentence that refers to the variable, for $g \times s = w$.
a. g. "g is a factor of w."
b. s. "s is a factor of w."
c. w. "w is a multiple of g (or s.)"

11. Tell whether each of the following is a prime or composite number. Explain how you know.
a. 31. It's a prime. To show this, we just have to show that no prime number less than $\sqrt{31}$ divides it. 2 does not divide 31, nor does 3, nor does 5. At this point we can stop, because the next prime is 7, and $7^2 = 49 > 31$. So 31 is prime.
b. $13 \times 23 = 299$. It's a composite because it has factors other than itself and 1 (13 and 23.)
c. 27. It's a composite because, for example, 3 divides it.
d. 999. It's a composite because, for example, 3 divides it.
e. $\frac{7}{11}$. The terms "prime" and "composite" refer only to integers > 1, so this is neither.

12. Is the set of all prime numbers closed under multiplication?
This is a nice little example. The answer is, most emphatically, NO. For the primes to be closed under multiplication, the product $p \times q$ of EVERY pair of primes p and q would have to be a prime. But in fact, there is NO pair of primes p, q such that $p \times q$ is prime: $p \times q$ is always composite, since p and q always divide it and $p \times q$ can't be p or q, since both p and q are > 1.

14. Find at least one example that shows the following conjecture is false. The sum of two prime numbers is an even number.
2 and 3 are prime, but $2 + 3 = 5$ which is odd.

3 Section 11.2

3. Make a factor tree and give the prime factorization.
It's kind of hard to type a factor tree, but I'll try.
a. $960 = 10 \times 96 = (5 \times 2) \times (12 \times 8) = (5 \times 2) \times ([4 \times 3] \times [4 \times 2]) = (5 \times 2) \times ([(2 \times 2) \times 3] \times [(2 \times 2) \times 2]$.
The leaves of our tree, the primes, are $5, 2, 2, 2, 3, 2, 2, 2$. So $960 = 2^6 \times 3 \times 5$.
e. 29. Since 29 is prime, the factor tree of 29 is just 29; the tree is just a single element, which is both root and leaf. And the prime factorization is simply 29.

4. Kim and Lee star their factor trees for 5000 differeently. Kim has brancehs to 5 and 1000, and Lee has brancehes to 50 and 100. Does this violate the Fundamental Theorem of Arithmetic? Explain.
No. The Fundamental Theorem of Arithmetic says that the *prime* factorization of a positive integer is unique. Neither 5×1000 nor 50×100 is a prime factorization of 5000. If Kim and Lee both continue factoring until they are left with only primes, the two factorizations they end up with will be the same.

7.Give (a) four prime factors and (b) six composite factors of
$3, 972, 672 = 2^6 \times 3 \times 19 \times 57$.
a. There are only four prime factors: $2, 3, 19$ and 57.
b. $2^2, 2^3, 2^4, 2^5, 2^6$ and 2×3 are six composite factors. By the way, there are $7 \times 2 \times 2 \times 2$ factors total.

8. Is there a whole number m such that $7^{15} = 9^m$? Explain.
No. If $m = 0$, 9^m is 1 (which 7^{15} is not), and if m is a whole number greater than 0, 9^m is a positive integer whose only prime factor is 3 (which 7^{15} is not, since its only prime factor is 7.)

9. Find the smallest number with the given factors.
a. 14,105 and 63.
We want the least commmon multiple, simple as that. So we find the prime factorizations. We have
$14 = 2 \times 7$
$105 = 3 \times 5 \times 7$,
$63 = 3^2 \times 7$.
The most 2's any of these has is 1.
The most 3's any of these has is 2.
The most 5's any of these has is 1.
The most 7's any of these has is 1.
There are no other prime factors in any of the three numbers.
So the least common multiple is $2 \times 3^2 \times 5 \times 7 = 630$.

11. Find, if possible, nonzero values for m and n such that $25^m = 125^n$.
In these sorts of problems, we should always look at the prime factorizations. We have $25 = 5^2$ and $125 = 5^3$. So we want m, n such that $(5^2)^m = (5^3)^n$, i.e.

such that
$5^{2m} = 5^{3n}$. Then we just want m and n such that $2m = 3n$. There are tons of pairs (m, n) that satisfy that equation; for example, $m = 3$ and $n = 2$, or $m = 9$ and $n = 6$.

4 Section 11.3

1. Which pairs of numbers are relatively prime? Explain.
A pair of numbers is relatively prime if their only common divisor is one. So they're relatively prime if their prime factorizations have no primes in common.
a. $27 = 3^3, 29 = 29$, these factorizations have no primes in common so the two numbers are relatively prime.
b. $25 = 5^2$, $35 = 5 \times 7$, these factorizations have 5 in common, so 25 and 35 are NOT relatively prime.

3. Practice the divisibility tests for 2,3,4,5,6,8,9 on these numbers.
a. $92, 236$.
2 divides 6 so it divides the number.
3 does not divide the sum of the digits $9+2+2+3+6 = 22$ so it doesn't divide the number.
4 divides the last two digits 36 so it divides the number.
5 clearly doesn't divide the number.
6 doesn't divide it because 3 doesn't divide it; 6 is a factor if and only if 2 and 3 are.
8 doesn't divide the last three digits 236 so it doesn't divide the number.
9 doesn't divide it because 3 doesn't divide it.

e. 2299.
2 doesn't divide it because it doesn't divide 9.
3 doesn't divide it because it doesn't divide the sum of the digits $2+2+9+9 = 22$.
4 doesn't divide it because 2 doesn't.
5 clearly doesn't.
6 doesn't because 2 doesn't.
8 doesn't because 2 doesn't.
9 doesn't because 3 doesn't.

h. 400. 2 divides it because 2 divides the last digit 0 (since $2 \times 0 = 0$.)
3 doesn't because it doesn't divide the sum of the digits.
4 does because it divides the last two digits 00; alternately, $400 = 4 \times 100$ so 4 divides it.
5 clearly divides it.
6 doesn't because 3 doesn't. 8 divides it because 8 divides the last 3 digits (which in this case is the whole number), 400.
9 doesn't because 3 doesn't.

4. In verifying that each of the following is a prime, what is the largest prime that must be checked as a possible factor?
To show that n is a prime, we just have to verify that no prime that is less than $\sqrt{n}$ divides it.
a. 401. Well, $20^2 = 400$, so $\sqrt{401}$ will be between 20 and 21. So we only have to check primes ≤ 20. The largest such is 19.
d. 1607. Pretty much the same logic, but with 40 instead of 20 (since $40^2 = 1600$ which is so close to 1607 that we can be sure 41^2 is > 1607. So we only have to check primes ≤ 40; the largest such is 37.

5. Which of the following are primes?
f. 119. Let's see. 2 doesn't divide it, nor does 3, nor 5. Maybe 7 does. Yeah, 7 does. $7 \times 17 = 119$, and both the factors are prime so that's the prime factorization. And a mighty ugly one, I might add.
h. 197. Let's see. 2 doesn't divide it, nor does 3, nor 5. Maybe 7 does. 7 divides $210, 203, 196$, nope. The next candidate is 11. 11 divides $220, 209, 198,$ *nope*. The next candidate is 13. We have $13^2 = 196$, so 13 doesn't divide it and we can stop now, since the square of the next prime will be > 197.
This painful procedure is, in general, the only means known to mankind for determining whether a given number is prime.

6. Give a 12-digit number that has 3 as a factor, but not 9, and 4 as a factor, but not 8.
Hmm. So we want the digits to sum to a multiple of 3, but not a multiple of 9. And we want 4 to divide the last two digits, but we DON'T want 8 to divide the last three.
So, the last three digits can be 012; 4 divides that, but not 8. And we want all the digits to add up to a number that has 3 as a factor, but not 9. So let's just stick a 3 out front.
$300, 000, 000, 012$. That works.

5 Section 11.4

3. The Earth takes 365 earth days, that is 1 year, to make one revoluation around the sun. It takes Saturn about 30 Earth years, Jupiter 12 Earth years, Mars 2 Earth years. If Earth, Jupiter, Saturn and Mars are all aligned, how long before they will all be aligned again?
We want the least common multiple of 1, 12, 2 and 30. The 1 is completely superfluous. The 2 is superfluous too, because 2 divides both 12 and 30 so it will divide any multiple of either. So we just want the least common multiple of 12 and 30. Multiples of 30 are easy to spot, so let's just count off the multiples of 12; $12, 24, 36, 48, 60$. There we are: 60 is the smallest multiple of 12 which is also a multiple of 30. So the answer is 60 years.