MA 22000 Lesson 20 Notes
(Calculus part of text) Section 4.4
Derivatives of Exponential Functions

Deriving the formula for a derivative of the function $f(x) = e^x$:

$$\frac{d(e^x)}{dx} = \lim_{h\to 0}\left(\frac{e^{x+h} - e^x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{e^x e^h - e^x}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{e^x(e^h - 1)}{h}\right)$$
$$= \lim_{h\to 0}\left[e^x\left(\frac{e^h - 1}{h}\right)\right]$$

For h values very close to zero, the value of $\left(\frac{e^h - 1}{h}\right)$ is approximately 1. (You can see a table on page 228 of your textbook to verify this.) Therefore:

$$\lim_{h\to 0}\left[e^x\left(\frac{e^h - 1}{h}\right)\right] = \lim_{h\to 0}(e^x \cdot 1)$$
$$\lim_{h\to 0}\left[e^x\left(\frac{e^h - 1}{h}\right)\right] = e^x$$

This argument leads to the following conclusion:

Derivative of e^x: $\frac{d}{dx}\left(e^x\right) = e^x$

**To find the rule to determine the derivative of exponential functions with bases other than *e*, we use the following fact. By the 'composition of inverse functions' in algebra, $e^{\ln a} = a$. Examine the following.

$e^{\ln a} = a$ by the definition of inverse functions

$e^{(\ln a)x} = \left(e^{(\ln a)}\right)^x$ Product rule of exponents

$= a^x$

Therefore: $a^x = e^{(\ln a)x}$

Use this fact to find the derivative of a^x.

Derivative of a^x:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right)$$
$$= (e^{(\ln a)x})(\ln a)$$
$$= (\ln a)a^x$$

$$\boxed{\frac{d}{dx}(a^x) = (\ln a)a^x}$$

****Note: There are no homework problems assigned with exponential functions with any base other than *e*. Therefore the above derivative rule is not necessary for students. I simply provide it for information purposes only.**

Remember the 'chain rule' must be used if the exponent is an expression (a function) other than the variable.

Derivatives of exponential functions (base *e*) using the chain rule:

$$\frac{d}{dx}\left(e^{g(x)}\right) = (e^{g(x)})(g'(x))$$

<u>Ex</u> 1: Find the derivatives:

$a)\quad y = e^{3x}$

$b)\quad y = e^{(5x+6x^2)}$

$c)\quad D_x(3.8e^{1.5x})$

$d)\quad h(x) = -12e^{x^2}$

$e) \quad y = 4e^{4x^2+2x}$

$f) \quad y = 3x^3e^{-4x}$

$g) \quad g(x) = (5x^2 - 7x^3)e^{-3x}$

$h) \quad y = \dfrac{e^{2x}}{3x^2+5}$

$i) \quad f(x) = (e^{4x^2} + 10x)^4$

$j) \quad y = \dfrac{200}{8+3e^x}$

Ex 2:
The quantity (in grams) of a radioactive substance present after t years is given by the model $Q(t) = 100e^{-0.421t}$.

a) Find the quantity when $t = 0$.

b) Find the rate of change after (a) 2 years, (b) 5 years.

Ex 3:
The growth of the world population (in millions) can be approximated by the function $A(t) = 3100e^{(0.0166t)}$ where t is the number of years since 1960. Find the instantaneous rate of change in the world population for the year (a) 2010 and (b) 2015.

Ex 4:
The amount (in grams) of a sample of an element present after *t* years is given by $A(t)=400e^{-0.4t}$. Find the rate of change of the quantity after (a) 3 years, (b) 5 years, and (c) 15 years.