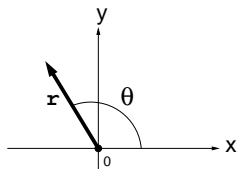


## Study Guide # 1

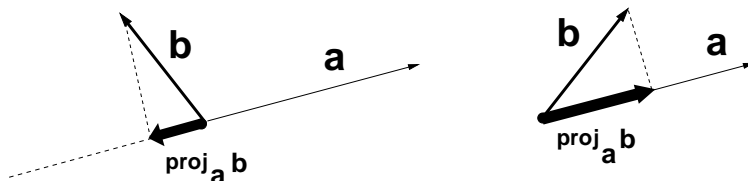
### 1. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

(a)  $\vec{v} = \langle a, b, c \rangle = a\vec{i} + b\vec{j} + c\vec{k}$ ; vector addition and subtraction geometrically using parallelograms spanned by  $\vec{u}$  and  $\vec{v}$ ; length or magnitude of  $\vec{v} = \langle a, b, c \rangle$ ,  $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$ ; directed vector from  $P_0(x_0, y_0, z_0)$  to  $P_1(x_1, y_1, z_1)$  given by  $\vec{v} = \overrightarrow{P_0P_1} = P_1 - P_0 = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$ .

(b) Dot (or inner) product of  $\vec{a} = \langle a_1, a_2, a_3 \rangle$  and  $\vec{b} = \langle b_1, b_2, b_3 \rangle$ :  $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ ; properties of dot product; useful identity:  $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ ; angle between two vectors  $\vec{a}$  and  $\vec{b}$ :  $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$ ;  $\vec{a} \perp \vec{b}$  if and only if  $\vec{a} \cdot \vec{b} = 0$ ; the vector in  $\mathbb{R}^2$  with length  $r$  with angle  $\theta$  is  $\vec{v} = \langle r \cos \theta, r \sin \theta \rangle$ :



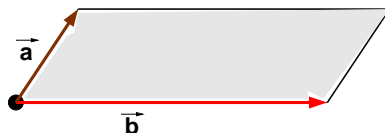
(c) Projection of  $\vec{b}$  along  $\vec{a}$ :  $\text{proj}_{\vec{a}} \vec{b} = \left\{ \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right\} \frac{\vec{a}}{|\vec{a}|}$ ; Work =  $\vec{F} \cdot \vec{D}$ .



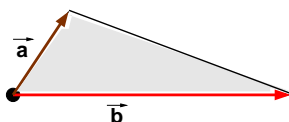
(d) Cross product (only for vectors in  $\mathbb{R}^3$ ):

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}$$

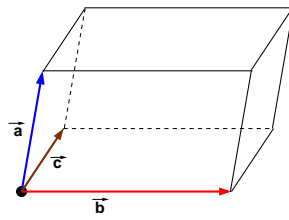
properties of cross products;  $\vec{a} \times \vec{b}$  is **perpendicular** (orthogonal or normal) to both  $\vec{a}$  and  $\vec{b}$ ; area of parallelogram spanned by  $\vec{a}$  and  $\vec{b}$  is  $A = |\vec{a} \times \vec{b}|$ :



the area of the triangle spanned is  $A = \frac{1}{2} |\vec{a} \times \vec{b}|$ :



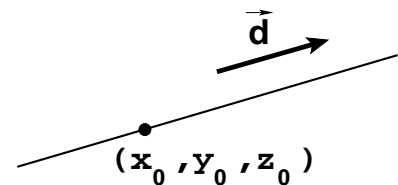
Volume of the parallelepiped spanned by  $\vec{a}, \vec{b}, \vec{c}$  is  $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$ :



2. Equation of a line  $L$  through  $P_0(x_0, y_0, z_0)$  with direction vector  $\vec{d} = \langle a, b, c \rangle$ :

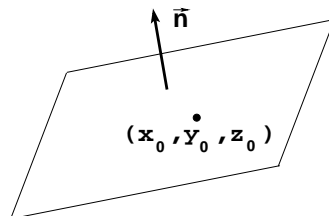
Vector Form:  $\vec{r}(t) = \langle x_0, y_0, z_0 \rangle + t \vec{d}$ .

Parametric Form: 
$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$



Symmetric Form:  $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ . (If say  $b = 0$ , then  $\frac{x - x_0}{a} = \frac{z - z_0}{c}$ ,  $y = y_0$ .)

3. Equation of the plane through the point  $P_0(x_0, y_0, z_0)$  and perpendicular to the vector  $\vec{n} = \langle a, b, c \rangle$  ( $\vec{n}$  is a *normal vector* to the plane) is  $\langle (x - x_0), (y - y_0), (z - z_0) \rangle \cdot \vec{n} = 0$ ; Sketching planes (consider  $x, y, z$  intercepts).



4. Quadric surfaces (can sketch them by considering various *traces*, i.e., curves resulting from the intersection of the surface with planes  $x = k$ ,  $y = k$  and/or  $z = k$ ); some generic equations have the form:

(a) *Ellipsoid*:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(b) *Elliptic Paraboloid*:  $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(c) *Hyperbolic Paraboloid (Saddle)*:  $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

(d) *Cone*:  $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

(e) *Hyperboloid of One Sheet*:  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

(f) *Hyperboloid of Two Sheets*:  $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

**5.** Vector-valued functions  $\vec{\mathbf{r}}(t) = \langle f(t), g(t), h(t) \rangle$ ; tangent vector  $\vec{\mathbf{r}}'(t)$  for smooth curves, unit tangent vector  $\vec{\mathbf{T}}(t) = \frac{\vec{\mathbf{r}}'(t)}{|\vec{\mathbf{r}}'(t)|}$ ; unit normal vector  $\vec{\mathbf{N}}(t) = \frac{\vec{\mathbf{T}}'(t)}{|\vec{\mathbf{T}}'(t)|}$  differentiation rules for vector functions, including:

(i)  $\{\phi(t) \vec{\mathbf{v}}(t)\}' = \phi(t) \vec{\mathbf{v}}'(t) + \phi'(t) \vec{\mathbf{v}}(t)$ , where  $\phi(t)$  is a real-valued function

(ii)  $(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})' = \vec{\mathbf{u}} \cdot \vec{\mathbf{v}}' + \vec{\mathbf{u}}' \cdot \vec{\mathbf{v}}$

(iii)  $(\vec{\mathbf{u}} \times \vec{\mathbf{v}})' = \vec{\mathbf{u}} \times \vec{\mathbf{v}}' + \vec{\mathbf{u}}' \times \vec{\mathbf{v}}$

(iv)  $\{\vec{\mathbf{v}}(\phi(t))\}' = \phi'(t) \vec{\mathbf{v}}'(\phi(t))$ , where  $\phi(t)$  is a real-valued function

**6.** Integrals of vector functions  $\int \vec{\mathbf{r}}(t) dt = \left\langle \int f(t) dt, \int g(t) dt, \int h(t) dt \right\rangle$ ; arc length of curve parameterized by  $\vec{\mathbf{r}}(t)$  is  $L = \int_a^b |\vec{\mathbf{r}}'(t)| dt$ ; arc length function  $s(t) = \int_a^t |\vec{\mathbf{r}}'(u)| du$ ; reparameterize by arc length:  $\vec{\sigma}(s) = \vec{\mathbf{r}}(t(s))$ , where  $t(s)$  is the inverse of the arc length function  $s(t)$ ; the *curvature* of a curve parameterized by  $\vec{\mathbf{r}}(t)$  is  $\kappa = \frac{|\vec{\mathbf{T}}'(t)|}{|\vec{\mathbf{r}}'(t)|}$ . **Note:**  $\sqrt{\alpha^2} = |\alpha|$ .

**7.**  $\vec{\mathbf{r}}(t)$  = position of a particle,  $\vec{\mathbf{r}}'(t) = \vec{\mathbf{v}}(t)$  = velocity;  $\vec{\mathbf{a}}(t) = \vec{\mathbf{v}}'(t) = \vec{\mathbf{r}}''(t)$  = acceleration;  $|\vec{\mathbf{r}}'(t)| = |\vec{\mathbf{v}}(t)|$  = speed; Newton's 2<sup>nd</sup> Law:  $\vec{\mathbf{F}} = m \vec{\mathbf{a}}$ .

**8.** Domain and range of a function  $f(x, y)$  and  $f(x, y, z)$ ; *level curves* (or contour curves) of  $f(x, y)$  are the curves  $f(x, y) = k$ ; using level curves to sketch surfaces; *level surfaces* of  $f(x, y, z)$  are the surfaces  $f(x, y, z) = k$ .

**10.** Partial derivatives  $\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$ ,

$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$ ; higher order derivatives:  $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ ,

$f_{yy} = \frac{\partial^2 f}{\partial y^2}$ ,  $f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$ , etc; mixed partials.

**11.** Equation of the tangent plane to the graph of  $z = f(x, y)$  at  $(x_0, y_0, z_0)$  is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

**12.** Total differential for  $z = f(x, y)$  is  $dz = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ ; total differential for  $w = f(x, y, z)$

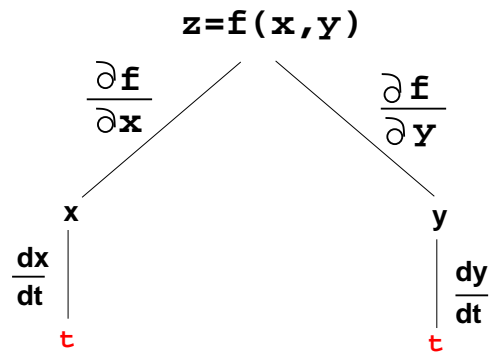
is  $dw = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$ ; linear approximation for  $z = f(x, y)$  is given by  $\Delta z \approx dz$ ,

i.e.,  $f(x + \Delta x, y + \Delta y) - f(x, y) \approx \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ , where  $\Delta x = dx$ ,  $\Delta y = dy$ ;

*Linearization* of  $f(x, y)$  at  $(a, b)$  is given by  $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$ ;  
 $L(x, y) \approx f(x, y)$  near  $(a, b)$ .

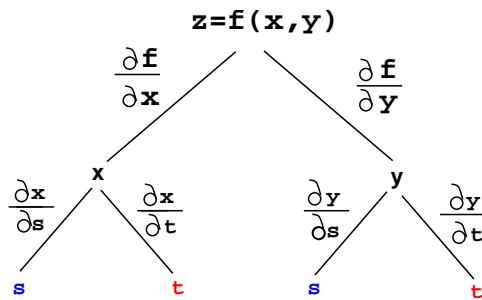
**13.** Different forms of the **Chain Rule**: Form 1, Form 2; General Form: Tree diagrams. For example:

(a) If  $z = f(x, y)$  and  $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ , then  $\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$  :



(b) If  $z = f(x, y)$  and  $\begin{cases} x = x(s, t) \\ y = y(s, t) \end{cases}$ , then

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} :$$



etc.....

## 14. Implicit Differentiation and Directional Derivative:

### Implicit Differentiation

Part I: If  $F(x, y) = 0$  defines  $y$  as function of  $x$  (i.e.,  $y = y(x)$ ), then to compute  $\frac{dy}{dx}$ , differentiate both sides of the equation  $F(x, y) = 0$  w.r.t.  $x$  and solve for  $\frac{dy}{dx}$ .

If  $F(x, y, z) = 0$  defines  $z$  as function of  $x$  and  $y$  (i.e.  $z = z(x, y)$ ), then to compute  $\frac{\partial z}{\partial x}$ , differentiate the equation  $F(x, y, z) = 0$  w.r.t.  $x$  (hold  $y$  fixed) and solve for  $\frac{\partial z}{\partial x}$ . For  $\frac{\partial z}{\partial y}$ , differentiate the equation  $F(x, y, z) = 0$  w.r.t.  $y$  (hold  $x$  fixed) and solve for  $\frac{\partial z}{\partial y}$ .

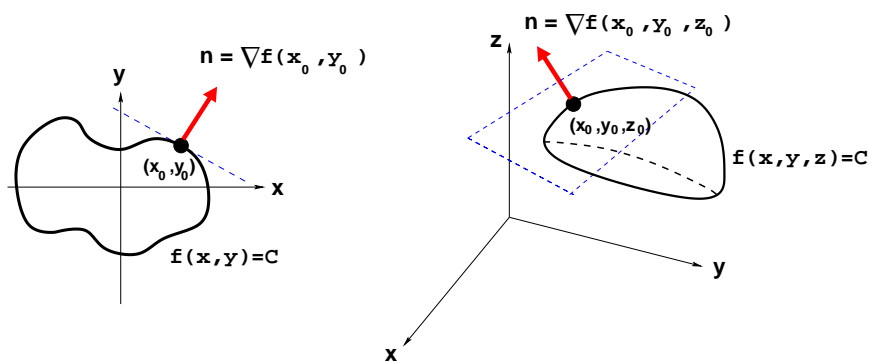
Part II: If  $F(x, y) = 0$  defines  $y$  as function of  $x \implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ ;

while if  $F(x, y, z) = 0$  defines  $z$  as function of  $x$  and  $y \implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$  and  $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$ .

### Directional derivative

Directional derivative of  $f(x, y)$  at  $(x_0, y_0)$  in the direction  $\vec{u}$ :  $D_{\vec{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$ , where  $\vec{u}$  must be a unit vector.

- 15.** Gradient vector for  $f(x, y)$ :  $\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ , properties of gradients; gradient points in direction of maximum rate of increase of  $f$ ; The maximum value of the directional derivative is equal to  $\|\nabla f\|$ ;  $\nabla f(x_0, y_0) \perp$  level curve  $f(x, y) = C$  and, in the case of 3 variables,  $\nabla f(x_0, y_0, z_0) \perp$  level surface  $f(x, y, z) = C$ :



- 16.** Relative/local extrema; critical points ( $\nabla f = \vec{0}$  or  $\nabla f$  does not exist);  $2^{nd}$  Derivatives Test: A critical points is a local min if  $D = f_{xx}f_{yy} - f_{xy}^2 > 0$  and  $f_{xx} > 0$ , local max if  $D > 0$  and  $f_{xx} < 0$ , saddle if  $D < 0$ ; absolute extrema; Max-Min Problems;