Study Guide # 2

1. Relative/local extrema; critical points ($\nabla f = \vec{0}$ or ∇f does not exist); 2nd Derivatives Test: A critical points is a local min if $D = f_{xx}f_{yy} - f_{xy}^2 > 0$ and $f_{xx} > 0$, local max if $D > 0$ and $f_{xx} < 0$, saddle if $D < 0$; absolute extrema; Max-Min Problems; **Lagrange Multipliers:** *Extremize* $f(\vec{x})$ subject to a constraint $g(\vec{x}) = C$, solve the system: $\nabla f = \lambda \nabla g$ and $g(\vec{x}) = C$.

2. Double integrals; Midpoint Rule for rectangle : $\displaystyle\iint_R f(x,y)\, dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\, \Delta A$;

3. Type I region $D : \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $D : \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\displaystyle\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$ and

$\displaystyle\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities: $mA \le \displaystyle\iint_D f(x,y)\, dA \le MA$, where $A =$ area of D and $m \le f(x,y) \le M$ on D.

5. Change of Variables Formula in Polar Coordinates: if $D : \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$\displaystyle\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, r\, dr\, d\theta.$
$\uparrow$

6. Applications of double integrals:

  (a) Area of region D is $A(D) = \displaystyle\iint_D dA$

  (b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $V = \displaystyle\iint_D f(x,y)\, dA$

  (c) Mass of D is $m = \displaystyle\iint_D \rho(x,y)\, dA$, where $\rho(x,y) =$ density (per unit area); sometimes write $m = \displaystyle\iint_D dm$, where $dm = \rho(x,y)\, dA$.

  (d) Moment about the x-axis $M_x = \displaystyle\iint_D y\,\rho(x,y)\, dA$; moment about the y-axis $M_y = \displaystyle\iint_D x\,\rho(x,y)\, dA$.

  (e) Center of mass $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\displaystyle\iint_D x\,\rho(x,y)\, dA}{\displaystyle\iint_D \rho(x,y)\, dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\displaystyle\iint_D y\,\rho(x,y)\, dA}{\displaystyle\iint_D \rho(x,y)\, dA}$

 <u>Remark</u>: centroid = center of mass when density is constant (this is useful).

7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E:

$$\iiint_E f(x,y,z)\, dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x,y,z)\, dz\, dA \text{ for } E = \{(x,y) \in D,\ u(x,y) \le z \le v(x,y)\};$$

volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.
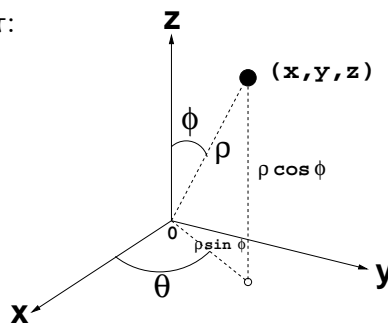
8. Cylindrical Coordinates (r, θ, z):

From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$
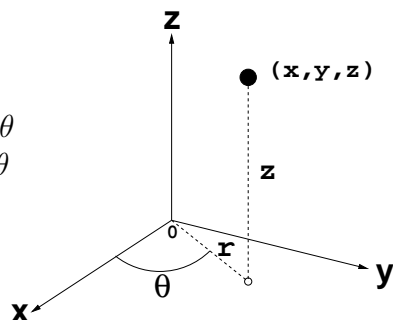
Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

9. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$

Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

10. Triple integrals in <u>Cylindrical Coordinates</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\, dz\, dr\, d\theta}$

$$\iiint_E f(x,y,z)\, dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

$\uparrow$

11. Triple integrals in <u>Spherical Coordinates</u>: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi\ d\rho\, d\phi\, d\theta}$

$$\iiint_E f(x,y,z)\, dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$\uparrow$

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\, Q(x,y) \rangle$ and $\vec{\mathbf{F}}(x,y,z) = \langle P(x,y),\, Q(x,y),\, R(x,y) \rangle$; $\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

13. Line integral of a function $f(x, y)$ along C, parameterized by $x = x(t), \quad y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>Remarks</u>:

(a) $\displaystyle\int_C f(x, y) \, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\displaystyle\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) \, x'(t) \, dt$

(c) $\displaystyle\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) \, y'(t) \, dt$

14. Line integral of vector field $\vec{\mathbf{F}}(x, y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) \, dt \,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

15. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}) \, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

16. If $\vec{\mathbf{F}}(x, y) = P(x, y) \, \vec{\mathbf{i}} + Q(x, y) \, \vec{\mathbf{j}}$, then $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x, y) \, dx + Q(x, y) \, dy$; Work $= \displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

17. If S is the graph of $z = f(x, y)$ above D, then the surface area of S is

$$A(S) = \iint_D \sqrt{1 + (\partial f / \partial x)^2 + (\partial f / \partial y)^2} \, dA.$$