Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

1. Line integral of a function $f(x,y,z)$ along C, parameterized by $x=x(t),\quad y=y(t),\quad z=z(t)$ and $a \le t \le b$, is

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),\,y(t),\,z(t))\,\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

Remarks:

(a) $\int_C f(x,y,z)\,ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y,z)\,dx = \int_a^b f(x(t),y(t),z(t))\,x'(t)\,dt$

(c) $\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,y'(t)\,dt$

(d) $\int_C f(x,y,z)\,dy = \int_a^b f(x(t),y(t),z(t))\,z'(t)\,dt$

2. Line integral of vector field $\vec{\mathbf{F}}(x,y,z)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t))\cdot\vec{\mathbf{r}}'(t)\,dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

3. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}}\,\cdot\,d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}}\cdot\vec{\mathbf{T}})\,ds$$

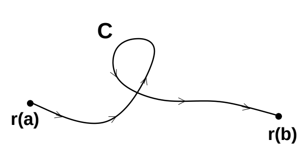
where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

4. If $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\,\vec{\mathbf{k}}$, then

$$\int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} = \int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz\,;$$

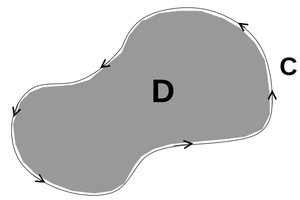
Work $= \int_C \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}}$.

5. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\int_C \nabla f\cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



6. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

7. <u>Green's Theorem</u>: $\displaystyle\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$ ($C =$ boundary of D):



As a consequence of Green's Theorem one has

$$\frac{1}{2}\int_C x\,dy - y\,dx = \int_C x\,dy = -\int_C y\,dx = Area(D)$$

8. Del Operator: $\dfrac{\partial}{\partial x}\,\vec{\mathbf{i}} + \dfrac{\partial}{\partial y}\,\vec{\mathbf{j}} + \dfrac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

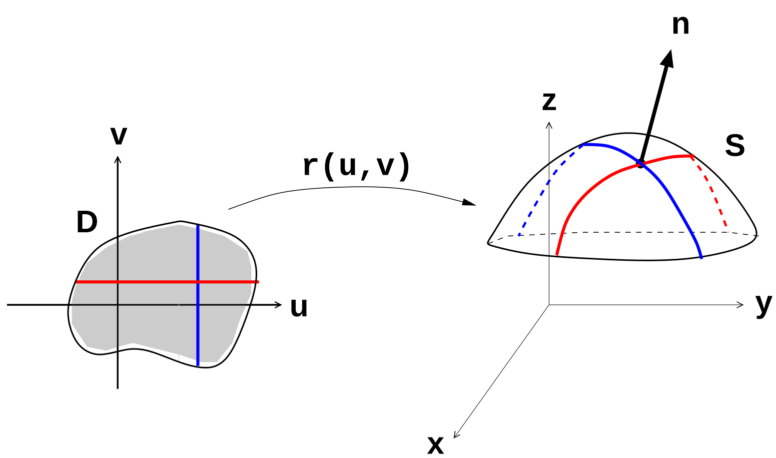
$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field (i.e., $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$).

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible*.

9. Parametric surface S: $\vec{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$:



Normal vector to surface S : $\vec{\mathbf{n}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v$; tangent planes and normal lines to parametric surfaces.

10. Surface area of a surface S:

(i) $A(S) = \iint_D |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dA$

(ii) If S is the graph of $z = h(x, y)$ above D, then $A(S) = \iint_D \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} \, dA$;

Remark: $dS = |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dA$ = differential of surface area; while $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) \, dA$

11. The surface integral of $f(x, y, z)$ over the surface S:

(i) $\iint_S f(x, y, z) \, dS = \iint_D f(\vec{\mathbf{r}}(u, v)) \, |\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v| \, dA.$

(ii) If S is the graph of $z = h(x, y)$ above D, then

$\iint_S f(x, y, z) \, dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} \, dA.$

12. The surface integral of $\vec{\mathbf{F}}$ over the surface S (recall, $d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) \, dA$):

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) \, dA.$$

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) \, dS = \iint_D \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) \, dA.$$

If S is the graph of $z = h(x, y)$ above D, with $\vec{\mathbf{n}}$ oriented upward, and $\vec{\mathbf{F}} = \langle P, Q, R \rangle$, then

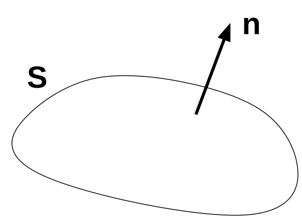
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \left(-P \frac{\partial h}{\partial x} - Q \frac{\partial h}{\partial y} + R \right) dA.$$

(i) Connection between surface integral of a vector field and a function:

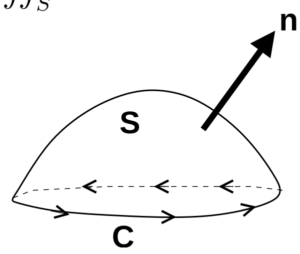
$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) \, dS.$$

(The above gives another way to compute $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$)

(ii) $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) \, dS =$ *flux* of $\vec{\mathbf{F}}$ across the surface S.



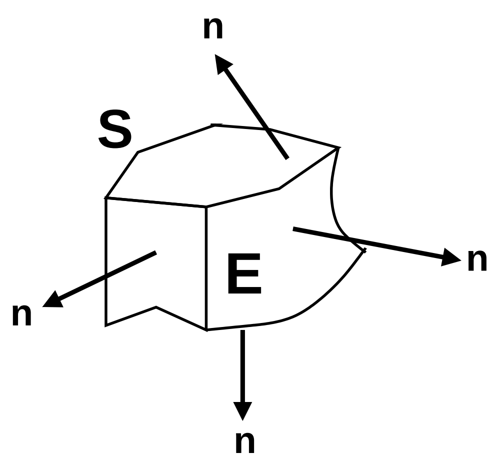
13. <u>Stokes' Theorem</u>: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$ (recall, $\operatorname{curl} \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}}$).



$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} =$ *circulation* of $\vec{\mathbf{F}}$ around C.

14. <u>The Divergence Theorem/Gauss' Theorem</u>: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_E \operatorname{div} \vec{\mathbf{F}} \; dV$

(recall, $\operatorname{div} \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}}$).



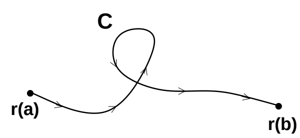
15. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t)$, where $a \le t \le b$	$S : \vec{\mathbf{r}}(u,v)$, where $(u,v) \in D$
$ds = \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt =$ differential of arc length	$dS = \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA =$ differential of surface area
$\int_C ds =$ length of C	$\iint_S dS =$ surface area of S
$\int_C f(x,y,z)\, ds = \int_a^b f(\vec{\mathbf{r}}(t))\, \lvert\vec{\mathbf{r}}\,'(t)\rvert\, dt$ (independent of orientation of C)	$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{\mathbf{r}}(u,v))\, \lvert\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v\rvert\, dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}\,'(t)\, dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}\,'(t)\, dt$ (depends on orientation of C)	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u,v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v)\ dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\right)\ ds$ The *circulation* of $\vec{\mathbf{F}}$ around C	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \left(\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\right)\ dS$ The *flux* of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

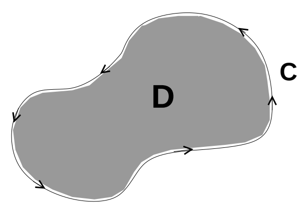
16. Integration Theorems:

<u>FUNDAMENTAL THEOREM OF CALCULUS</u>: $\int_a^b F'(x)\,dx = F(b) - F(a)$

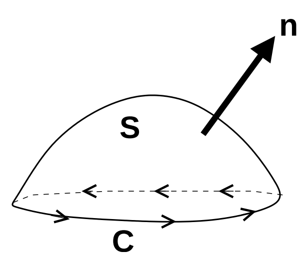
<u>FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS</u>: $\int_a^b \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$



<u>GREEN'S THEOREM</u>: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$



<u>STOKES' THEOREM</u>: $\iint_S \operatorname{curl} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$



<u>DIVERGENCE THEOREM</u>: $\iiint_E \operatorname{div} \vec{\mathbf{F}} \; dV = \iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$