CHAPTER 4
Exponential and Logarithmic Functions

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Section 4.1 Exponential Functions

Solutions to Even-Numbered Exercises

2. (a) $\left(\frac{1}{5}\right)^3 = \frac{1}{125}$

(b) $\left(\frac{1}{8}\right)^{1/3} = \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

(c) $64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16$

(d) $\left(\frac{5}{8}\right)^2 = \frac{25}{64}$

(e) $100^{3/2} = \left(\sqrt{100}\right)^3 = 10^3 = 1000$

(f) $4^{5/2} = \left(\sqrt{4}\right)^5 = 2^5 = 32$

4. (a) $\frac{5^3}{5^6} = \frac{1}{5^{6-3}} = \frac{1}{5^3} = \frac{1}{125}$

(b) $\left(\frac{1}{5}\right)^{-2} = \left(\frac{5}{1}\right)^2 = 25$

(c) $(8^{1/2})(2^{1/2}) = (8 \cdot 2)^{1/2} = 16^{1/2} = \sqrt{16} = 4$

(d) $(32^{3/2})\left(\frac{1}{2}\right)^{3/2} = \left(32 \cdot \frac{1}{2}\right)^{3/2}$

$= 16^{3/2} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

6. (a) $(4^3)(4^2) = (64)(16) = 1024$

(b) $\left(\frac{1}{4}\right)^2(4^2) = \left(\frac{1}{4} \cdot 4\right)^2 = (1)^2 = 1$

(c) $(4^6)^{1/2} = 4^3 = 64$

(d) $[(8^{-1})(8^{2/3})]^3 = (8^{-1/3})^3 = 8^{-1} = \frac{1}{8}$

8. $f(x) = 3^{x+2}$

(a) $f(-4) = 3^{-4+2} = \frac{1}{9}$

(b) $f\left(-\frac{1}{2}\right) = 3^{-1/2+2} = 3^{3/2} \approx 5.196$

(c) $f(2) = 3^{2+2} = 81$

(d) $f\left(-\frac{5}{2}\right) = 3^{-5/2+2} = 3^{-1/2} = \frac{1}{\sqrt{3}} \approx 0.577$

10. $g(x) = 1.075^x$

(a) $g(1.2) \approx 1.091$

(b) $g(180) \approx 450{,}322.416$

(c) $g(60) \approx 76.649$

(d) $g(12.5) \approx 2.469$

12. $5^{x+1} = 125$

$5^{x+1} = 5^3$

$x + 1 = 3$

$x = 2$

14. $\left(\frac{1}{5}\right)^{2x} = 625$

$5^{-2x} = 5^4$

$-2x = 4$

$x = -2$

16. $4^2 = (x+2)^2$

$\pm 4 = x + 2$

$x = 2, -6$

18. $(x+3)^{4/3} = 16$

$x + 3 = 16^{3/4}$

$x + 3 = 8$

$x = 5$

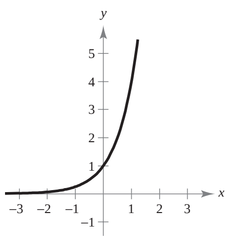
20. The graph of $f(x) = 3^{-x/2} = (1/3)^{x/2}$ is an exponential curve with the following characteristics.
Passes through $(0, 1)$, $\left(1, 1/\sqrt{3}\right)$, $(2, 1/3)$
Horizontal asymptote: $y = 0$
Therefore, it matches graph (c).

22. The graph of $f(x) = 3^{x-2}$ is an exponential curve with the following characteristics.
Passes through $(0, 1/9)$, $(2, 1)$, $(3, 3)$
Horizontal asymptote: $y = 0$
Therefore, it matches graph (f).

24. The graph of $f(x) = 3^x + 2$ is an exponential curve with the following characteristics.
Passes through $(0, 3)$, $(1, 5)$, $(-1, 7/3)$
Horizontal asymptote: $y = 2$
Therefore, it matches graph (b).

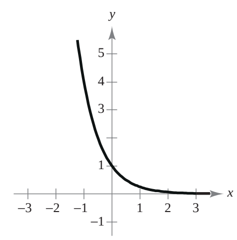
26. $f(x) = 4^x$

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16



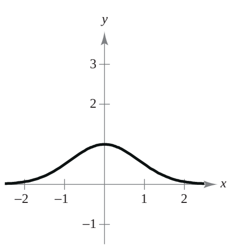
28. $f(x) = \left(\frac{1}{4}\right)^x$

x	-2	-1	0	1	2
$f(x)$	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$



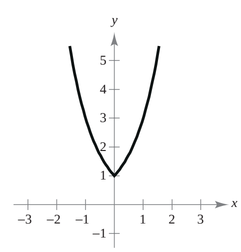
30. $y = 2^{-x^2}$

x	-2	-1	0	1	2
y	$\frac{1}{16}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{16}$



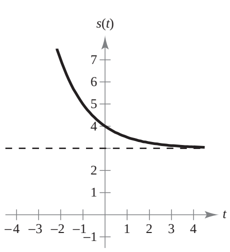
32. $y = 3^{|x|}$

x	-2	-1	0	1	2
y	9	3	1	3	9



34. $s(t) = 2^{-t} + 3 = \left(\frac{1}{2}\right)^t + 3$

t	-2	-1	0	1	2
$s(t)$	7	5	4	$\frac{7}{2}$	$\frac{13}{4}$



36. $S(t) = 116.59(1.3295)^t$ ($t = 4$ corresponds to 1994)

(a) 2006: $S(16) = 116.59(1.3295)^{16} \approx \$11{,}109$ million

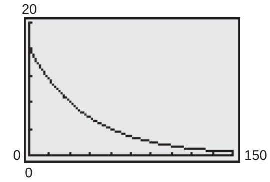
(b) 2012: $S(22) = 116.59(1.3295)^{22} \approx \$61{,}349$ million

38. $C(t) = P(1.05)^t, \quad 0 \le t \le 10$

$C(10) = 6.95(1.05)^{10} \approx \$11.32$

40. $y = 16\left(\frac{1}{2}\right)^{t/30}, \quad t \ge 0$

(a)



(b) 50 years: $y = 16\left(\frac{1}{2}\right)^{50/30} \approx 5.04$ grams

(c) $y = 16\left(\frac{1}{2}\right)^{t/30} = 1 \Longrightarrow t = 120$ years.

Section 4.2 Natural Exponential Functions

2. (a) $\left(\dfrac{1}{e}\right)^{-2} = e^2$

(b) $\left(\dfrac{e^5}{e^2}\right)^{-1} = (e^3)^{-1} = \dfrac{1}{e^3}$

(c) $\dfrac{e^5}{e^3} = e^2$

(d) $\dfrac{1}{e^{-3}} = e^3$

4. (a) $(e^{-3})^{2/3} = e^{-2} = \dfrac{1}{e^2}$

(b) $\dfrac{e^4}{e^{-1/2}} = e^{4+1/2} = e^{9/2} = e^4\sqrt{e}$

(c) $(e^{-2})^{-4} = e^8$

(d) $(e^{-4})(e^{-3/2}) = e^6$

6. $e^x = 1 = e^0$

$x = 0$

8. $e^{-1/x} = \sqrt{e} = e^{1/2}$

$-\dfrac{1}{x} = \dfrac{1}{2}$

$x = -2$

10. $\dfrac{x^2}{2} = e^2$

$x^2 = 2e^2$

$x = \pm\sqrt{2}e$

12. $x^{-2} = \dfrac{2}{e^2}$

$x^2 = \dfrac{e^2}{2}$

$x = \pm\dfrac{e}{\sqrt{2}} = \pm\dfrac{\sqrt{2}e}{2}$

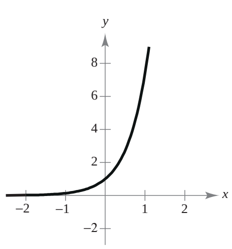
14. $f(x) = e^{-x/2}$. Decaying exponential. Matches (e)

16. $f(x) = e^{-1/x}$. Matches (b)

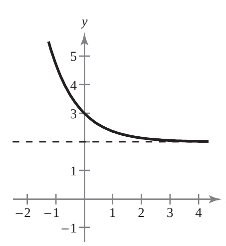
18. $f(x) = e^{-x} + 1$. Matches (a)

20. $f(x) = e^{2x}$

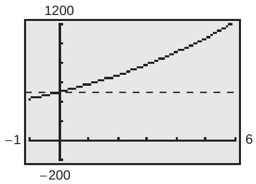
x	-1	0	$\frac{1}{2}$	1	2
$f(x)$	0.135	1	2.718	7.389	54.598



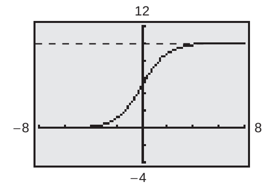
22. $f(x) = e^{-x+2}$. Intercept $(0, 7.4)$.
Decaying exponential.



24. $A(t) = 500e^{0.15t}$

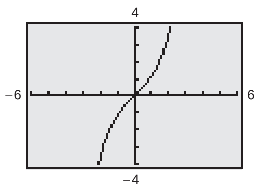


26. $g(x) = \dfrac{10}{1 + e^{-x}}$



28.

x	-2	-1	0	1	2
y	-3.627	-1.175	0	1.175	3.627



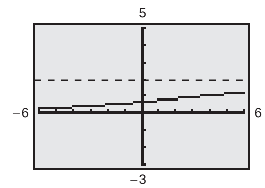
$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \frac{\infty - 0}{2} = \infty$$

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = \frac{0 - \infty}{2} = -\infty$$

No horizontal asymptotes
Continuous on the entire real line

30.

x	-2	-1	0	1	2
y	0.502	0.581	0.667	0.758	0.854



$$\lim_{x\to\infty} \frac{2}{1 + 2e^{-0.2x}} = \frac{2}{1 + 0} = 2$$

$$\lim_{x\to-\infty} \frac{2}{1 + 2e^{-0.2x}} = \frac{2}{\infty} = 0$$

Horizontal asymptotes: $y = 0$ and $y = 2$
Continuous on the entire real line

32. $A = P\left(1 + \dfrac{r}{n}\right)^{nt}, \quad r = 0.05, t = 20, P = 2500$

$$= 2500\left(1 + \frac{0.05}{n}\right)^{20n}$$

Continuous compounding: $A = Pe^{rt} = 2500e^{(0.05)20}$

n	1	2	4	12	365	continuous
A	6633.24	6712.66	6753.71	6781.60	6795.24	6795.70

34. $A = P\left(1 + \dfrac{r}{n}\right)^{nt}, \quad P = 2500, r = 0.05, t = 40$

$$= 2500\left(1 + \frac{0.05}{n}\right)^{40n}$$

Continuous compounding: $A = Pe^{rt} = 2500e^{(0.05)(40)}$

n	1	2	4	12	365	continuous
A	17,599.97	18,023.92	18,245.05	18,396.04	18,470.11	18,472.64

36. $A = Pe^{rt}, A = 100{,}000, r = 0.03 \Longrightarrow P = 100{,}000e^{-0.03t}$

t	1	10	20	30	40	50
P	97,044.55	74,081.82	54,881.16	40,656.97	30,119.42	22,313.02

38. $A = P\left(1 + \frac{r}{n}\right)^{nt}$, $A = 100{,}000$, $r = 0.06$, $n = 365 \Longrightarrow P = \frac{100{,}000}{\left(1 + \frac{0.06}{365}\right)^{365t}}$

t	1	10	20	30	40	50
P	94,176.92	54,883.87	30,112.39	16,532.33	9073.58	4979.93

40. $\sqrt{eff} = \left(1 + \frac{r}{n}\right)^n - 1, \quad r = 0.075$

(a) $\sqrt{eff} = \left(1 + \frac{0.075}{1}\right)^1 - 1 = 0.075$ or 7.5%

(b) $\sqrt{eff} = \left(1 + \frac{0.075}{2}\right)^2 - 1 \approx 0.0764$ or 7.64%

(c) $\sqrt{eff} = \left(1 + \frac{0.075}{4}\right)^4 - 1 \approx 0.0771$ or 7.71%

(d) $\sqrt{eff} = \left(1 + \frac{0.075}{12}\right)^{12} - 1 \approx 0.0776$ or 7.76%

42. $P = \frac{A}{\left(1 + \frac{r}{n}\right)^{nt}} = \frac{21{,}154.03}{\left(1 + \frac{0.078}{12}\right)^{(12)(4)}} \approx \$15{,}500.00$

44. $A = 6000\left(1 + \frac{0.0625}{12}\right)^{(12)(3)} \approx \$7233.86$

46. (a) $p(1000) = 10{,}000\left(1 - \frac{3}{3 + 1000e^{-0.001(1000)}}\right)$

$= \$9919.11$

(b) $p(1500) = 10{,}000\left(1 - \frac{3}{3 + 1500e^{-0.001(1500)}}\right)$

$= \$9911.16$

48.

s	50	55	60	65	70
y	28.0	26.4	24.8	23.4	22.0

You can conclude that the miles per gallon decreases as speed increases.

50. $P = 115.49e^{0.0445t}$ ($t = 0$ corresponds to 1970)

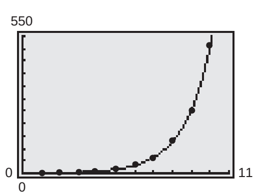
(a) 1970: $P(0) = 115.49e^{0.0445(0)} \approx 115.49$ thousand
1980: $P(10) = 115.49e^{0.0445(10)} \approx 180.22$ thousand
1990: $P(20) \approx 281.23$ thousand
2000: $P(30) \approx 438.86$ thousand

(b) The population is growing exponentially, not linearly.

(c) Using a graphing utility, $t \approx 42$, or 2012.

52.

Interval	1	2	3	4	5	6	7	8	9	10
Number of cells	1	2	4	8	16	32	64	128	256	512



Model: $y = 2^{t-1}$ or $y = e^{(t-1)\ln 2}$

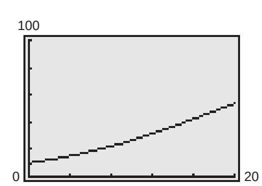
54. $N = \frac{95}{1 + 8.5e^{-0.12t}}$

(a) $t = 10 \Longrightarrow N = 26.68$ words/min

(b) $N = 70 \Longrightarrow t \approx 26.4$ weeks

(c) Yes, there is a limit as t increases without bound.

$$\lim_{t\to\infty} \frac{95}{1 + 8.5e^{-0.12t}} = \frac{95}{1 + 0} = 95 \text{ words/min}$$



Section 4.3 Derivatives of Exponential Functions

2. $y' = 2e^{2x}$

$y'(0) = 2$

4. $y' = -2e^{-2x}$

$y'(0) = -2$

6. $y' = -e^{1-x}$

8. $f(x) = e^{1/x} = e^{x^{-1}}$

$$f'(x) = e^{1/x}\left(-\frac{1}{x^2}\right) = \frac{-e^{1/x}}{x^2}$$

10. $g(x) = e^{\sqrt{x}} = e^{x^{1/2}}$

$$g'(x) = e^{\sqrt{x}}\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

12. $y = 4x^3e^{-x}$

$$y' = 4x^3(-e^{-x}) + 12x^2e^{-x}$$
$$= 4x^2e^{-x}(3 - x)$$

14. $f(x) = \dfrac{(e^x + e^{-x})^4}{2}$

$$f'(x) = 2(e^x + e^{-x})^3(e^x - e^{-x})$$

16. $y' = x^2e^x + 2xe^x - 2xe^x - 2e^x + 2e^x = x^2e^x$

18. $g'(x) = e^{x^3}(3x^2)$

$g'(-1) = 3e^{-1}$

$$y - \frac{1}{e} = \frac{3}{e}(x + 1)$$
$$y = \frac{3}{e}x + \frac{3}{e} + \frac{1}{e} = \frac{3}{e}x + \frac{4}{e}$$

20. $y = \dfrac{x}{e^{2x}} = xe^{-2x}, \quad \left(1, \dfrac{1}{e^2}\right)$

$$y' = x(-2e^{-2x}) + e^{-2x}$$
$$y'(1) = \frac{-2}{e^2} + \frac{1}{e^2} = \frac{-1}{e^2}$$
$$y - \frac{1}{e^2} = \frac{-1}{e^2}(x - 1)$$
$$ye^2 - 1 = -x + 1$$
$$ye^2 + x - 2 = 0 \quad \text{or} \quad y = \frac{2 - x}{e^2}$$

22. $y = (e^{4x} - 2)^2 \quad (0, 1)$

$$y' = 2(e^{4x} - 2)(4e^{4x})$$
$$y'(0) = 2(-1)4 = -8$$
$$y - 1 = -8(x - 0)$$
$$y = -8x + 1$$

24. $x^2y - xe^x + 2 = 0$

$$2xy + x^2\frac{dy}{dx} - xe^x - e^x = 0$$
$$x^2\frac{dy}{dx} = xe^x + e^x - 2xy$$
$$\frac{dy}{dx} = \frac{xe^x + e^x - 2xy}{x^2}$$

26. $$e^{xy} + x^2 - y^2 = 10$$

$$\left(y + x\frac{dy}{dx}\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$

$$\frac{dy}{dx} = \frac{-ye^{xy} - 2x}{xe^{xy} - 2y} = \frac{-y(10 - x^2 + y^2) - 2x}{x(10 - x^2 + y^2) - 2y} = \frac{x^2y - y^3 - 2x - 10y}{xy^2 - x^3 + 10x - 2y}$$

28. $f'(x) = (1 + 2x)(4e^{4x}) + 2e^{4x} = 2e^{4x}[(1 + 2x)(2) + 1] = 2e^{4x}(4x + 3)$

$f''(x) = 2e^{4x}(4) + 8e^{4x}(4x + 3) = 8e^{4x}[1 + (4x + 3)] = 32e^{4x}(x + 1)$

30. $f'(x) = (3 + 2x)(-3e^{-3x}) + 2e^{-3x} = e^{-3x}(-9 - 6x + 2) = -e^{-3x}(6x + 7)$

$f''(x) = -e^{-3x}(6) + (6x + 7)(3e^{-3x}) = 3e^{-3x}(-2 + 6x + 7) = 3e^{-3x}(6x + 5)$

32. $f(x) = \frac{1}{2}(e^x - e^{-x})$

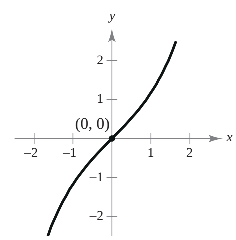
$f'(x) = \frac{1}{2}(e^x + e^{-x}) > 0$ for all x

$f'(x) \neq 0$ for any values of x. Thus, there are no relative extrema.

$f''(x) = \frac{1}{2}(e^x - e^{-x})$

$f''(x) = 0$ when $x = 0$.

We have a point of inflection at $(0, 0)$.



x	-2	-1	0	1	2
$f(x)$	-3.627	-1.175	0	1.175	3.627

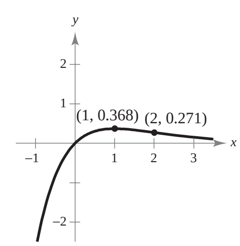
34. $f(x) = xe^{-x}$

$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1 - x)$

$f'(x) = 0$ when $x = 1$.

$f''(x) = e^{-x}(-1) + (1 - x)(-e^{-x}) = -e^{-x}[1 + (1 - x)] = e^{-x}(x - 2)$

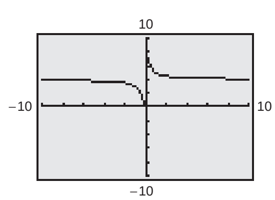
Since $f''(1) < 0$, we have a relative maximum at $(1, e^{-1})$.
$f''(x) = 0$ when $x = 2$ and we have a point of inflection at $(2, 2e^{-2})$.



x	-1	0	1	2	3
$f(x)$	-2.718	0	0.368	0.271	0.149

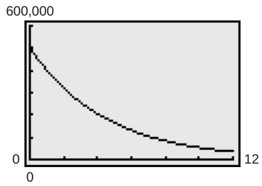
36. $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

Horizontal asymptote: $y = 4$
Vertical asymptote: $x = 0$ (from the left)



38. (a) $V = 500{,}000e^{-0.2231t}$

$V' = -111{,}500e^{-0.2231t}$



(b) $V'(1) = -111{,}500e^{-0.2231(1)} \approx -\$89{,}243.89$ per year

(c) $V'(5) = -111{,}500e^{-0.2231(5)} \approx -\$36{,}650.66$ per year

(d) $V(0) = 500{,}000,\ V(10) \approx 53{,}710.5$

$$\frac{V(10) - V(0)}{10 - 0} = \frac{53{,}710.5 - 500{,}000}{10} \approx -44{,}629$$

Linear model: $V - 500{,}000 = -44{,}629(t - 0)$

$V = 500{,}000 - 44{,}629t$

(e) Answers will vary.

40. $N = \dfrac{95}{1 + 8.5e^{-0.12t}}, \qquad N' = \dfrac{96.9e^{-0.12t}}{(1 + 8.5e^{-0.12t})^2}$

(a) When $t = 5$, $N' = 1.66$ words/min/week.

(b) When $t = 10$, $N' = 2.30$ words/min/week.

(c) When $t = 30$, $N' = 1.74$ words/min/week.

42. $a = 20, \quad b = 0.5$

$p = 80e^{-0.5t} + 20$

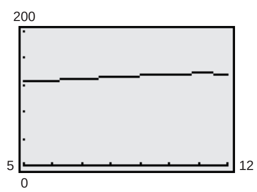
$\dfrac{dp}{dt} = -40e^{-0.5t}$

(a) When $t = 1$, $dp/dt \approx -24.3\%$.

(b) When $t = 3$, $dp/dt \approx -8.9\%$.

44. $y = 115.46 + 1.592t + 0.0552t^2 - 0.00004e^t$ ($t = 5$ corresponds to 1995)

(a)



(b) 1995: $y'(5) \approx 2.14$ million per year

1998: $y'(8) \approx 2.36$ million per year

2002: $y'(12) \approx -3.59$ million per year

(c) $y' = 1.592 + 0.1104t - 0.00004e^t$

$y'(5) \approx 2.14$

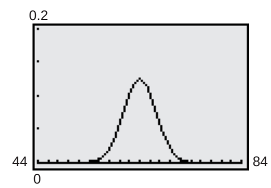
$y'(8) \approx 2.36$

$y'(12) \approx -3.59$

46. $\mu = 64,\ \sigma = 3.2$

(a) $f(x) = \dfrac{1}{3.2\sqrt{2\pi}} e^{-(x-64)^2/20.48}$

(b)



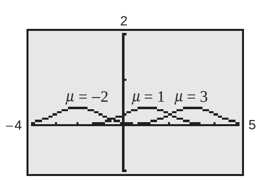
(c) $f'(x) = \dfrac{1}{3.2\sqrt{2\pi}} e^{-(x-64)^2/20.48}[-2(x - 64)/20.48]$

$= \dfrac{-1}{32.768\sqrt{2\pi}}(x - 64)e^{-(x-64)^2/20.48}$

(d) For $x < \mu = 64$, $f'(x) > 0$, and for $x > \mu = 64$, $f'(x) < 0$.

48. $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} = \dfrac{1}{\sqrt{2\pi}} e^{-(x-\mu)^2/2}$

μ shifts the graph horizontally.



Section 4.4 Logarithmic Functions

2. $e^{2.128\ldots} = 8.4$

4. $e^{-2.2882\ldots} = 0.056$

6. $\ln 7.389\ldots = 2$

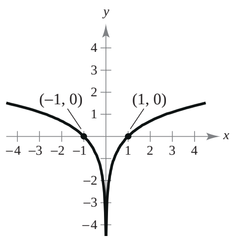
8. $\ln 1.284\ldots = 0.25$

10. The graph is a logarithmic curve that passes through the point $(1, 0)$ with a vertical asymptote at $x = 0$. Therefore, it matches graph (d).

12. The graph is a logarithmic curve that passes through the point $(2, 0)$ with a vertical asymptote at $x = 1$. Therefore, it matches graph (a).

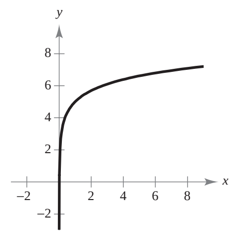
14.

x	± 0.5	± 1	± 2
y	-0.69	0	0.69



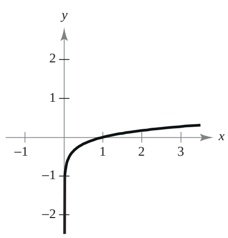
16.

x	0.5	1	2	3	4
y	4.31	5	5.69	6.10	6.39



18.

x	0.5	1	2	3	4
y	-0.17	0	0.17	0.27	0.35

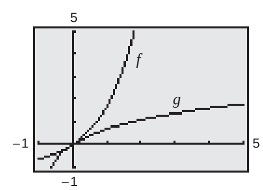


20. $f(g(x)) = f(\ln(x+1))$

$= e^{\ln(x+1)} - 1 = (x+1) - 1 = x$

$g(f(x)) = g(e^x - 1)$

$= \ln((e^x - 1) + 1) = \ln e^x = x$

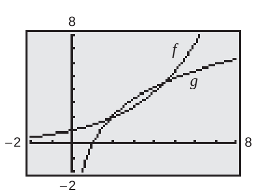


22. $f(g(x)) = f(\ln x^3)$

$= f(3 \ln x) = e^{3 \ln x/3} = e^{\ln x} = x$

$g(f(x)) = g(e^{x/3})$

$= \ln(e^{x/3})^3 = \ln e^x = x$



24. $\ln e^{2x-1} = 2x - 1$

26. $e^{\ln \sqrt{x}} = \sqrt{x}$

28. $-8 + e^{\ln x^3} = -8 + x^3$

$= x^3 - 8$

30. (a) $\ln 0.25 = \ln \frac{1}{4} = \ln 1 - \ln 4 = \ln 1 - \ln 2^2 = \ln 1 - 2\ln 2 = 0 - 2(0.6931) = -1.3862$

(b) $\ln 24 = \ln(3 \cdot 2^3) = \ln 3 + 3\ln 2 = 1.0986 + 3(0.6931) = 3.1779$

(c) $\ln \sqrt[3]{12} = \frac{1}{3}\ln(3 \cdot 2^2) = \frac{1}{3}[\ln 3 + 2\ln 2] = \frac{1}{3}[1.0986 + 2(0.6931)] \approx 0.8283$

(d) $\ln \frac{1}{72} = \ln 1 - \ln 72 = 0 - \ln(2^3 \cdot 3^2) = -[3\ln 2 + 2\ln 3] = -[3(0.6931) + 2(1.0986)] = -4.2765$

32. $\ln \dfrac{1}{5} = \ln 1 - \ln 5 = 0 - \ln 5 = -\ln 5$

34. $\ln \dfrac{xy}{z} = \ln xy - \ln z = \ln x + \ln y - \ln z$

36. $\ln \sqrt{\dfrac{x^3}{x+1}} = \dfrac{1}{2}[\ln x^3 - \ln(x+1)] = \dfrac{3}{2}\ln x - \dfrac{1}{2}\ln(x+1)$

38. $\ln\left[x\sqrt[3]{x^2+1}\right] = \ln x + \ln(x^2+1)^{1/3} = \ln x + \dfrac{1}{3}\ln(x^2+1)$

40.
$$\begin{aligned}\ln \frac{2x}{\sqrt{x^2-1}} &= \ln 2x - \ln\sqrt{x^2-1}\\ &= \ln 2 + \ln x - \frac{1}{2}\ln[(x+1)(x-1)]\\ &= \ln 2 + \ln x - \frac{1}{2}[\ln(x+1) + \ln(x-1)]\\ &= \ln 2 + \ln x - \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x-1)\end{aligned}$$

42.
$$\begin{aligned}\ln(2x+1) + \ln(2x-1) &= \ln(2x+1)(2x-1)\\ &= \ln(4x^2-1)\end{aligned}$$

44. $2\ln 3 - \dfrac{1}{2}\ln(x^2+1) = \ln 9 - \ln\sqrt{x^2+1} = \ln \dfrac{9}{\sqrt{x^2+1}}$

46. $\dfrac{1}{3}[2\ln(x+3) + \ln x - \ln(x^2-1)] = \dfrac{1}{3}\ln\dfrac{(x+3)^2x}{(x^2-1)} = \ln\sqrt[3]{\dfrac{x(x+3)^2}{x^2-1}}$

48.
$$\begin{aligned}2\left[\ln x + \tfrac{1}{4}\ln(x+1)\right] &= 2\ln x + \tfrac{1}{2}\ln(x+1)\\ &= \ln x^2 + \ln(x+1)^{1/2}\\ &= \ln[x^2(x+1)^{1/2}]\end{aligned}$$

50. $\frac{1}{2}\ln(x-1) + \frac{3}{2}\ln(x+2) = \ln(x-1)^{1/2} + \ln(x+2)^{3/2} = \ln\left[(x-1)^{1/2}(x+2)^{3/2}\right]$

52.
$$\begin{aligned}e^{\ln x^2} - 9 &= 0\\ x^2 - 9 &= 0\\ x^2 &= 9\\ x &= \pm 3\end{aligned}$$

54.
$$\begin{aligned}2\ln x &= 4\\ \ln x &= 2\\ x &= e^2\end{aligned}$$

56.
$$\begin{aligned}e^{-0.5x} &= 0.075\\ -0.5x &= \ln 0.075\\ x &= \frac{\ln 0.075}{-0.5}\\ &\approx 5.1805\end{aligned}$$

58. $400e^{-0.0174t} = 1000$

$$e^{-0.0174t} = \frac{1000}{400} = \frac{5}{2} = 2.5$$

$$-0.0174t = \ln 2.5$$

$$t = \frac{\ln 2.5}{-0.0174} \approx -52.66$$

60. $2e^{-x+1} - 5 = 9$

$$e^{-x+1} = 7$$

$$-x + 1 = \ln 7$$

$$x = 1 - \ln 7 \approx -0.9459$$

62. $\dfrac{50}{1 + 12e^{-0.02x}} = 10.5$

$$1 + 12e^{-0.02x} = \frac{50}{10.5}$$

$$e^{-0.02x} \approx 0.3135$$

$$-0.02x \approx \ln(0.3135)$$

$$x \approx -50 \ln(0.3135) \approx 57.9991$$

64. $2^{1-x} = 6$

$$\ln 2^{1-x} = \ln 6$$

$$(1 - x) \ln 2 = \ln 6$$

$$1 - x = \frac{\ln 6}{\ln 2}$$

$$1 - \frac{\ln 6}{\ln 2} = x$$

$$x \approx -1.5850$$

66. $400(1.06)^t = 1300$

$$(1.06)^t = \frac{13}{4}$$

$$t \ln 1.06 = \ln 13 - \ln 4$$

$$t = \frac{\ln 13 - \ln 4}{\ln 1.06}$$

$$\approx 20.2279$$

68. $2000\left(1 + \dfrac{0.06}{12}\right)^{12t} = 10{,}000$

$$\left(1 + \frac{0.06}{12}\right)^{12t} = \frac{10{,}000}{2000} = 5$$

$$12t \ln(1.005) = \ln 5$$

$$t = \frac{\ln 5}{12 \ln(1.005)} \approx 26.891$$

70. $3P = Pe^{rt}$

$$3 = e^{rt}$$

$$\ln 3 = rt$$

$$t = \frac{\ln 3}{r}$$

r	2%	4%	6%	8%	10%	12%	14%
t	54.93	27.47	18.31	13.73	10.99	9.16	7.85

72. $p = 250 - 0.8e^{0.005x}$

(a) $p = 200 = 250 - 0.8e^{0.005x}$

$$0.8e^{0.005x} = 50$$

$$e^{0.005x} = 62.5$$

$$0.005x = \ln 62.5$$

$$x = \frac{\ln 62.5}{0.005} \approx 827.03 \approx 827 \text{ units}$$

(b) $p = 125 = 250 - 0.8e^{0.005x}$

$$0.8e^{0.005x} = 125$$

$$e^{0.005x} = 156.25$$

$$0.005x = \ln 156.25$$

$$x = \frac{\ln 156.25}{0.005} \approx 1010.29 \approx 1010 \text{ units}$$

74. $P = 2734.07e^{0.0210t}$ ($t = 0$ corresponds to 1980)

(a) 2000: $P(20) = 2734.07e^{0.0210(20)} \approx 4161$ thousand

(b) $2734.07e^{0.0210t} = 6000$

$$e^{0.021t} = 2.19453$$

$$0.021t = \ln(2.19453)$$

$$t \approx \frac{\ln(2.19453)}{0.021} \approx 37.4, \text{ or } 2017.$$

76. $0.27 \times 10^{-12} = 10^{-12}\left(\frac{1}{2}\right)^{t/5715}$

$$0.27 = \left(\frac{1}{2}\right)^{t/5715}$$

$$\ln 0.27 = \frac{t}{5715}\ln\frac{1}{2}$$

$$t = \frac{5715 \ln 0.27}{\ln 1/2} \approx 10{,}795 \text{ year}$$

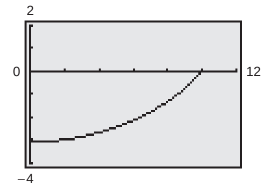
78. $0.13 \times 10^{-12} = 10^{-12}\left(\frac{1}{2}\right)^{t/5715}$

$$0.13 = \left(\frac{1}{2}\right)^{t/5715}$$

$$\ln 0.13 = \frac{t}{5715}\ln\frac{1}{2}$$

$$t = \frac{5715 \ln 0.13}{\ln 1/2} \approx 16{,}822 \text{ years}$$

80.



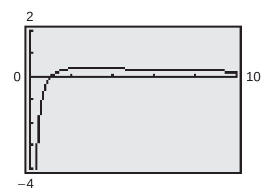
Answers will vary.

82. $f(x) = \dfrac{\ln x}{x}$

x	1	5	10	10^2	10^4	10^6
$f(x)$	0	0.3219	0.2303	0.0461	0.0009	0.00001

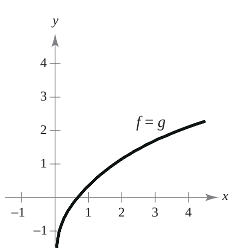
(a) $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$

(b)



Relative maximum at (2.7183, 0.3679)
No relative minima

84.



The graphs appear to be identical.

86. True. $\ln(ax) = \ln a + \ln x$

88. False. $\frac{1}{2}f(x) = \frac{1}{2}\ln x = \ln x^{1/2} \neq \sqrt{\ln x}$

90. True. $\ln x < 0$ for $0 < x < 1$.

Section 4.5 Derivatives of Logarithmic Functions

2. $y = \ln x^{5/2} = \frac{5}{2}\ln x$

$y' = \frac{5}{2x}$

$y'(1) = \frac{5}{2}$

4. $y = \ln x^{1/2} = \frac{1}{2}\ln x$

$y' = \frac{1}{2x}$

$y'(1) = \frac{1}{2}$

6. $f(x) = \ln 2x$

$f'(x) = \frac{2}{2x} = \frac{1}{x}$

8. $y = \ln(1 - x^2)$

$y' = -\frac{2x}{1 - x^2} = \frac{2x}{x^2 - 1}$

10. $y = \ln(1 - x)^{3/2} = \frac{3}{2}\ln(1 - x)$

$y' = \frac{3}{2}\left(\frac{-1}{1 - x}\right) = \frac{3}{2(x - 1)}$

12. $y = (\ln x^2)^2 = (2\ln x)^2 = 4(\ln x)^2$

$y' = 4\left[2(\ln x)\left(\frac{1}{x}\right)\right] = \frac{8\ln x}{x}$

14. $y = \frac{\ln x}{x^2}$

$y' = \frac{x^2(1/x) - 2x\ln x}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}$

16. $y = \ln\frac{x}{x^2 + 1} = \ln x - \ln(x^2 + 1)$

$y' = \frac{1}{x} - \frac{1}{x^2 + 1}(2x) = \frac{1}{x} - \frac{2x}{x^2 + 1} = \frac{1 - x^2}{x(x^2 + 1)}$

18. $y = \ln\frac{x^2}{x^2 + 1} = \ln x^2 - \ln(x^2 + 1)$

$y' = \frac{2}{x} - \frac{2x}{x^2 + 1} = \frac{2}{x(x^2 + 1)}$

20. $y = \ln\sqrt{\frac{x + 1}{x - 1}} = \frac{1}{2}[\ln(x + 1) - \ln(x - 1)]$

$y' = \frac{1}{2}\left[\frac{1}{x + 1} - \frac{1}{x - 1}\right] = \frac{-1}{x^2 - 1} = \frac{1}{1 - x^2}$

22. $y = \ln\left(x\sqrt{4 + x^2}\right) = \ln x + \ln(4 + x^2)^{1/2}$

$y' = \frac{1}{x} + \frac{x}{4 + x^2} = \frac{4 + 2x^2}{x(4 + x^2)}$

24. $f(x) = x\ln e^{x^2} = x(x^2) = x^3$

$f'(x) = 3x^2$

26. $f(x) = \ln\frac{1 + e^x}{1 - e^x} = \ln(1 + e^x) - \ln(1 - e^x)$

$f'(x) = \frac{e^x}{1 + e^x} - \frac{-e^x}{1 - e^x} = \frac{e^x(1 - e^x) + e^x(1 + e^x)}{(1 + e^x)(1 - e^x)} = \frac{2e^x}{1 - e^{2x}}$

28. $3^x = e^{x(\ln 3)}$

30. $\log_3 x = \frac{1}{\ln 3}\ln x$

32. $\log_5 12 = \frac{1}{\ln 5}\ln 12 \approx 1.544$ (calculator)

34. $\log_7\left(\frac{2}{9}\right) = \frac{\ln(2/9)}{\ln(7)} \approx -0.773$

36. $\log_{2/3} 32 = \frac{\ln 32}{\ln(2/3)} \approx -8.548$

38. $y = \left(\frac{1}{4}\right)^x$

$y' = \left(\ln\frac{1}{4}\right)\left(\frac{1}{4}\right)^x = (-\ln 4)\left(\frac{1}{4}\right)^x$

40. $g(x) = \log_5 x$

$g'(x) = \frac{1}{\ln 5}\cdot\frac{1}{x} = \frac{1}{x\ln 5}$

42. $y = 6^{5x}$

$y' = \ln 6 \cdot 6^{5x}(5) = 5\ln 6 \cdot 6^{5x}$

44. $f(x) = 10^{x^2}$

$f'(x) = (\ln 10)10^{x^2}(2x) = 2x(\ln 10)10^{x^2}$

46. $y = x3^{x+1}$

$y' = 3^{x+1} + x3^{x+1}\ln 3$

$= 3^{x+1}[1 + x\ln 3]$

48. $y' = \dfrac{x(1/x) - \ln x}{x^2} = \dfrac{1 - \ln x}{x^2}$

$y'(e) = 0$

$y - \dfrac{1}{e} = 0(x - e)$

$y = \dfrac{1}{e}$ Tangent line

50. $g(x) = \log_2(3x - 1) = \dfrac{\ln(3x - 1)}{\ln 2}$

$g'(x) = \dfrac{3}{(3x - 1)\ln 2}$

$g'(11) = \dfrac{3}{32\ln 2}$

$y - 5 = \dfrac{3}{32\ln 2}(x - 11)$

$y = \dfrac{3}{32\ln 2}x + 5 - \dfrac{33}{32\ln 2}$

52. $\ln xy + 5x = 30$

$\ln x + \ln y + 5x = 30$

$\dfrac{1}{x} + \dfrac{1}{y}\dfrac{dy}{dx} + 5 = 0$

$\dfrac{dy}{dx} = \left(-\dfrac{1}{x} - 5\right)y$

$= -\dfrac{y}{x} - 5y = -\dfrac{y(1 + 5x)}{x}$

54. $4xy + \ln(x^2y) = 7$

$4xy + 2\ln x + \ln y = 7$

$4xy' + 4y + \dfrac{2}{x} + \dfrac{y'}{y} = 0$

$\left(4x + \dfrac{1}{y}\right)y' = -4y - \dfrac{2}{x}$

$y' = \dfrac{-4y - (2/x)}{4x + (1/y)} = \dfrac{-4xy^2 - 2y}{4x^2y + x}$

56. $f(x) = 2\ln x + 3$

$f'(x) = \dfrac{2}{x}$

$f''(x) = -\dfrac{2}{x^2}$

58. $f(x) = \log_{10} x$

$f'(x) = \dfrac{1}{\ln 10} \cdot \dfrac{1}{x} = \dfrac{1}{\ln 10}x^{-1}$

$f''(x) = \dfrac{-1}{\ln 10}x^{-2} = \dfrac{-1}{x^2\ln 10}$

60. $T = 87.97 + 34.96\ln p + 7.91\sqrt{p}$

$\dfrac{dT}{dp} = \dfrac{34.96}{p} + \dfrac{7.91}{2\sqrt{p}}$

At $p = 60$, $\dfrac{dT}{dp} = \dfrac{34.96}{60} + \dfrac{7.91}{2\sqrt{60}} \approx 1.093$ degrees per pounds per square inch.

62. $f(x) = 2\ln x^3 = 6\ln x$

$f'(x) = \dfrac{6}{x}$

At $(e, 6)$, the slope of the tangent line is $f'(e) = 6/e$.

Tangent line: $y - 6 = \dfrac{6}{e}(x - e)$

$y = \dfrac{6}{e}x$

64. $f(x) = \ln\left(x\sqrt{x + 3}\right) = \ln x + \dfrac{1}{2}\ln(x + 3)$

$f'(x) = \dfrac{1}{x} + \dfrac{1}{2}\left(\dfrac{1}{x + 3}\right) = \dfrac{3(x + 2)}{2x(x + 3)}$

At $(1.20, 0.90)$, the slope of the tangent line is $f'(1.20) = \frac{20}{21}$.

Tangent line: $y - 0.90 = \dfrac{20}{21}(x - 1.20)$

$21y - 18.90 = 20x - 24$

$0 = 20x - 21y - 5.10$

$0 = 200x - 210y - 51$

66. $f(x) = x^2 \log_3 x = x^2 \dfrac{\ln x}{\ln 3}$

$$f'(x) = \frac{1}{\ln 3}(x + 2x \ln x)$$

$$f'(1) = \frac{1}{\ln 3}$$

$$y - 0 = \frac{1}{\ln 3}(x - 1)$$

$$y = \frac{x}{\ln 3} - \frac{1}{\ln 3}$$

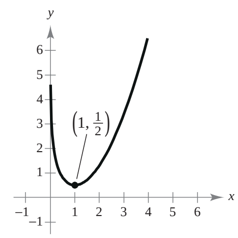
68. $y = \dfrac{x^2}{2} - \ln x$

$$y' = x - \frac{1}{x} = \frac{x^2 - 1}{x}$$

$y' = 0$ when $x = 1$.

$$y'' = 1 + \frac{1}{x^2} = \frac{x^2 + 1}{x^2}$$

[**Note:** $x = -1$ is not in the domain of the function.] Since $y''(1) = 2 > 0$, there is a relative minimum at $\left(1, \frac{1}{2}\right)$. Moreover, since $y'' > 0$ on $(0, \infty)$, it follows that the graph is concave upward in its domain and there are no inflection points.



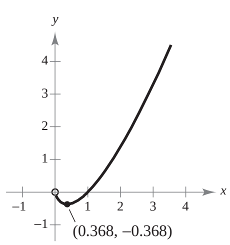
70. $y = x \ln x$

$$y' = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$

$y' = 0$ when $x = e^{-1}$.

$$y'' = \frac{1}{x}$$

Since $y''(e^{-1}) > 0$, there is a relative minimum at $(e^{-1}, -e^{-1})$. Moreover, since $y'' > 0$ on $(0, \infty)$, it follows that the graph is concave upward in its domain and there are no inflection points.



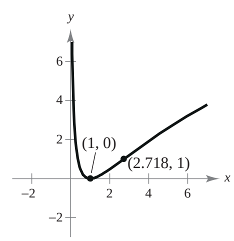
72. $y = (\ln x)^2$

$$y' = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2 \ln x}{x}$$

$y' = 0$ when $x = 1$.

$$y'' = \frac{x(2/x) - (2 \ln x)(1)}{x^2} = \frac{2(1 - \ln x)}{x^2}$$

Since $y''(1) > 0$, there is a relative minimum at $(1, 0)$. Since $y'' = 0$ when $x = e$, it follows that there is an inflection point at $(e, 1)$.



74. $x = 1000 - p \ln p$

$$\frac{dx}{dp} = 0 - \left[p\left(\frac{1}{p}\right) + (\ln p)(1)\right] = -(1 + \ln p)$$

When $p = 10$, $\dfrac{dx}{dp} = -(1 + \ln 10) \approx -3.3$.

76. $x = 300 - 50 \ln(\ln p)$

$$\frac{dx}{dp} = 0 - 50\frac{1/p}{\ln p} = -\frac{50}{p \ln p}$$

When $p = 10$, $\dfrac{dx}{dp} = \dfrac{-50}{10 \ln 10} \approx -2.171$.

78. $x = \dfrac{500}{\ln(p^2 + 1)}$

$$\ln(p^2 + 1) = \frac{500}{x}$$

$$p^2 + 1 = e^{500/x}$$

$$p = \sqrt{e^{500/x} - 1}$$

$$\frac{dp}{dx} = \frac{1}{2}(e^{500/x} - 1)^{-1/2}(e^{500/x})\left(\frac{-500}{x^2}\right)$$

$$= \frac{-250}{x^2} \cdot \frac{e^{500/x}}{\sqrt{e^{500/x} - 1}}$$

When $p = 10$,

$$x = \frac{500}{\ln(101)} \approx 108.34 \text{ and } \frac{dp}{dx} = -0.215.$$

Note that $dp/dx = 1/[dx/dp]$ from Exercise 75.

80. $C = 100 + 25x - 120 \ln x,\ x \geq 1$

(a) $\overline{C} = \dfrac{C}{x} = \dfrac{100}{x} + 25 - 120\dfrac{\ln x}{x}$

(b) $\overline{C}' = \dfrac{-100}{x^2} - 120\left[\dfrac{x(1/x) - \ln x}{x^2}\right]$

$$= \frac{-100}{x^2} - \frac{120}{x^2}(1 - \ln x)$$

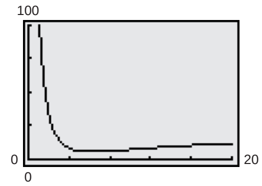
Setting $\overline{C}' = 0$,

$$100 = -120(1 - \ln x)$$

$$\frac{5}{6} = \ln x - 1$$

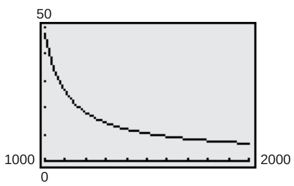
$$\frac{11}{6} = \ln x$$

$$x = e^{11/6} \approx 6.25,\ \overline{C}(6.25) \approx 5.81.$$

(c) 

Minimum of 5.81 at $x = 6.25$.

82. $t = \dfrac{5.315}{\ln x - 6.7968}, \quad x > 1000$

(a) 

(b) If $x = 1167.41$, $t = 20$ and the total amount is $(1167.41)(20)(12) \approx \$280,178$.

(c) If $x = 1068.45$, $t = 30$ and the total amount is $(1068.45)(30)(12) \approx \$384,642$.

(d) $t' = \dfrac{(\ln x - 6.7968)(0) - 5.315(1/x)}{(\ln x - 6.7968)^2}$

$$= \frac{-5.315/x}{(\ln x - 6.7968)^2}$$

$$= \frac{-5.315}{x(\ln x - 6.7968)^2}$$

When $x = 1167.41$, the instantaneous rate of change is

$$\frac{-5.315}{1167.41(\ln 1167.41 - 6.7968)^2} \approx -0.064.$$

When $x = 1068.45$, the instantaneous rate of change is

$$\frac{-5.315}{1068.45(\ln 1068.45 - 6.7968)^2} \approx -0.158.$$

(e) For a higher monthly payment, the term is shorter and the overall payment total smaller.

Section 4.6 Exponential Growth and Decay

2. Since $y = \frac{1}{2}$ when $t = 0$, it follows that $C = \frac{1}{2}$. Moreover, since $y = 5$ when $t = 5$, we have $5 = \frac{1}{2}e^{5k}$ and

$$k = \frac{1}{5}\ln 10 \approx 0.4605.$$

Thus, $y = \frac{1}{2}e^{0.4605t}$.

4. Since $y = 2$ when $t = 0$, it follows that $C = 2$. Moreover, since $y = 1$ when $t = 5$, we have $1 = 2e^{5k}$ and

$$k = \frac{1}{5}\ln\frac{1}{2} \approx -0.1386.$$

Thus, $y = 2e^{-0.1386t}$.

6. Using the fact that $y = \frac{1}{2}$ when $t = 3$ and $y = 5$ when $t = 4$, we have $\frac{1}{2} = Ce^{3k}$ and $5 = Ce^{4k}$. From these two equations we have

$$\frac{1/2}{e^{3k}} = \frac{5}{e^{4k}},\ \frac{1}{2}e^{4k} = 5e^{3k}, \text{ and } e^k = 10.$$

Thus, $k = \ln 10$ and we have $y = Ce^{t\ln 10}$. Since $5 = Ce^{4\ln 10}$, it follows that

$$C = \frac{5}{e^{4\ln 10}} = \frac{1}{2000}.$$

Therefore,

$$y = \frac{1}{2000}e^{t\ln 10} \approx \frac{1}{2000}e^{2.3026t}.$$

8. $\dfrac{dy}{dt} = -\dfrac{2}{3}y$

$y = 20$ when $t = 0$: $y = 20e^{-(2/3)t}$

$$\frac{dy}{dt} = 20\left(-\frac{2}{3}\right)e^{-(2/3)t} = -\frac{2}{3}\left[20e^{-(2/3)t}\right] = -\frac{2}{3}y$$

Exponential decay

10. $\dfrac{dy}{dt} = 5.2y$

$y = 18$ when $t = 0$: $y = 18e^{5.2t}$

$$\frac{dy}{dt} = 18(5.2)e^{5.2t} = 5.2(18e^{5.2t}) = 5.2y$$

Exponential growth

12. From Example 1 we have

$$y = Ce^{kt} = Ce^{[\ln(1/2)/1599]t}$$

$$1.5 = Ce^{[\ln(1/2)/1599]1000} \implies C \approx 2.314.$$

The initial quantity is 2.314 grams.

When $t = 10{,}000$,

$$y = 2.314e^{[\ln(1/2)/1599]10{,}000} \approx 0.03 \text{ grams.}$$

14. Since $y = Ce^{k}t = 3e^{[\ln(1/2)/5715]t}$, we have

$t = 1000$: $y = 3e^{[\ln(1/2)/5715)1000} \approx 2.66$ grams

$t = 10{,}000$: $y = 3e^{[\ln(1/2)/5715]10{,}000} \approx 0.89$ grams.

16. Since $y = Ce^{kt} = Ce^{[\ln(1/2)/24{,}100]t}$, we have

$$0.4 = Ce^{[\ln(1/2)/24{,}100]10{,}000} \implies C \approx 0.533.$$

The initial quantity is 0.533 grams.

When $t = 1000$,

$$y = 0.533e^{[\ln(1/2)/24{,}100]1000} \approx 0.518.$$

18. $0.9957C = Ce^{\ln(1/2)/h}$

when $t = 1$ and h is the half-life

$$0.9957 = e^{\ln(1/2)/h}$$

$$\ln(0.9957) = \frac{\ln\left(\frac{1}{2}\right)}{h}$$

$$h = \frac{\ln\left(\frac{1}{2}\right)}{\ln(0.9957)}$$

$$h \approx 160.85 \text{ years}$$

20. $0.30C = Ce^{[\ln(1/2)/5715]t}$

$$\ln 0.30 = \frac{\ln(1/2)}{5715}t$$

$$t = \frac{5715\ln 0.30}{\ln(1/2)} \approx 9927 \text{ years}$$

22. $(0, 8), \left(20, \frac{1}{2}\right)$

$y_1 = 8e^{k_1 t}$

$\frac{1}{2} = 8e^{20k_1} \Longrightarrow k_1 = \frac{1}{20}\ln\left(\frac{1}{16}\right) = \frac{-\ln 16}{20} \approx -0.1386$

$y_1 = 8e^{-0.1386t}$

$y_2 = 8(2)^{k_2 t}$

$\frac{1}{2} = 8(2)^{20k_2} \Longrightarrow k_2 = \frac{1}{20}\log_2\left(\frac{1}{16}\right) = \frac{-\ln 16}{20 \ln 2} \approx -0.2$

$y_2 = 8(2)^{-0.2t}$

$k_1 = (\ln 2)k_2$

24. (a) Let $t = 0$ represent 1960.

$y = Ce^{kt} = 2.3^{kt}$

$12 = 2.3e^{k(40)} \Longrightarrow 40k = \ln\left(\frac{12}{2.3}\right) \Longrightarrow k \approx 0.0413$

$y = 2.3e^{0.0413t}$ Exponential model

1970: $y(10) = 2.3e^{0.0413(10)} \approx 3.5$ million

1980: $y(20) \approx 5.3$ million

1900: $y(30) \approx 7.9$ million

(b) $24 = 2.3e^{0.0413t}$

$t = \frac{1}{0.0413}\ln\left(\frac{24}{2.3}\right) \approx 56.8$ years

(c) Increasing by 0.0413, or 4.13% each year.

26. Since $A = 20{,}000e^{0.105t}$, the time to double is given by the following.

$40{,}000 = 20{,}000e^{0.105t}$

$\ln 2 = 0.105t$

$t = \frac{\ln 2}{0.105} \approx 6.601$ years

Amount after 10 years:

$A = 20{,}000e^{0.105(10)} \approx \$57{,}153.02$

Amount after 25 years:

$A = 20{,}000e^{0.105(25)} \approx \$276{,}091.48$

28. Since $A = 10{,}000e^{rt}$ and $A = 20{,}000$ when $t = 5$, we have the following.

$20{,}000 = 10{,}000e^{5r}$

$r = \frac{\ln 2}{5} \approx 0.1386 = 13.86\%$

Amount after 10 years:

$A = 10{,}000e^{[(\ln 2)/5](10)} = \$40{,}000$

Amount after 25 years:

$A = 10{,}000e^{[(\ln 2)/5](25)} = \$320{,}000$

30. Since $A = 2000e^{rt}$ and $A = 6008.33$ when $t = 25$, we have the following.

$6008.33 = 2000e^{25r}$

$\ln 3.004165 = 25r$

$r \approx 0.044 = 4.4\%$

The time to double is given by

$4000 = 2000e^{0.044t}$

$\ln 2 = 0.044t$

$t \approx 15.753$ years.

Amount after 10 years:

$A = 2000e^{0.044(10)} \approx \$3105.41$

32. (a) By equating $A = P(1 + i)^t$ and $A = Pe^{rt}$ we have the following.

$P(1 + i)^t = Pe^{rt}$

$(1 + i)^t = (e^r)^t$

$1 + i = e^r$

Therefore, $i = e^r - 1$.

(b) If $r = 0.06$, then $i = e^{0.06} - 1 \approx 0.0618$ or 6.18%.

34.

Number of Compoundings per Year	4	12	365	Continuous
Effective Yield	7.714%	7.763%	7.788%	7.788%

$n = 4$: $i = \left(1 + \frac{0.075}{4}\right)^4 - 1 \approx 0.07714 \approx 7.714\%$

$n = 12$: $i = \left(1 + \frac{0.075}{12}\right)^{12} - 1 \approx 0.07763 \approx 7.763\%$

$n = 365$: $i = \left(1 + \frac{0.075}{365}\right)^{365} - 1 \approx 0.07788 \approx 7.788\%$

Continuous: $i = e^{0.075} - 1 \approx 0.07788 \approx 7.788\%$

36. (a) For $r = 10\%$, the approximate time necessary for the investment to double is $\frac{70}{10} = 7$ years.

(b) For $r = 7\%$, the approximate time necessary for the investment to double is $\frac{70}{7} = 10$ years.

38. (a) Let $t = 0$ correspond to 1990.

Date points: (0, 150), (10, 1074)

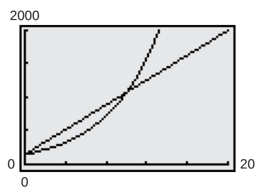
$y = 150(1.21756)^t = 150e^{0.19685t}$ Exponential models

$y = 92.4t + 150$ Linear model

(b) 2006: $y = 150e^{0.19685(16)} \approx \$3499$ million

(c) 2006: $y = 92.4(16) + 150 \approx \$1628.4$ million

(d)



Answers will vary.

40. $S = 30(1 - 3^{kt})$

(a) Since $S = 5$ when $t = 1$, we have

$5 = 30(1 - e^k)$

$e^k = 1 - \frac{1}{6}$

$k = \ln \frac{5}{6}$.

[**Note:** S is in thousands of units.]
Therefore, $S = 30(1 - e^{[\ln(5/6)]t})$.

(b) $\lim_{t\to\infty} 30(1 - e^{[\ln(5/6)]t}) = 30$

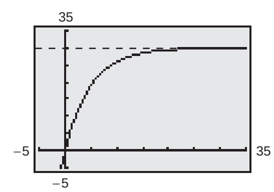
The saturation point for the market is 30,000 units.

(c) When $t = 5$, we have

$S = 30(1 - e^{[\ln(5/6)](5)}) \approx 17.944$.

Thus, $S \approx 17{,}944$ units.

(d)



42. (a) Since $N = 20$ when $t = 30$, we have

$$20 = 30(1 - e^{30k})$$

$$k = \frac{\ln(1/3)}{30} \approx -0.0366$$

$$N = 30(1 - e^{-0.0366t}).$$

(b) $25 = 30(1 - e^{-0.0366t})$

$$t = \frac{\ln(1/6)}{-0.0366} \approx 49 \text{ days}$$

44. (a) Since $p = Ce^{kx}$ where $p = 45$ when $x = 1000$ and $p = 40$ when $x = 1200$, we have the following.

$$45 = Ce^{1000k} \text{ and } 40 = Ce^{1200k}$$
$$\ln 45 = \ln C + 1000k$$
$$\ln 40 = \ln C + 1200k$$
$$\ln 45 - \ln 40 = -200k$$
$$k = \frac{\ln(45/40)}{-200} \approx -0.0005889$$

Therefore, we have $45 = Ce^{1000(-0.0005889)}$ which implies that $C \approx 81.0915$ and $p = 81.0915e^{-0.0005889x}$.

(b) Since $R = xp = 81.0915xe^{-0.0005889x}$, we have the following.

$$R' = 81.0915[-0.0005889xe^{-0.0005889x} + e^{-0.0005889x}]$$
$$= 81.0915e^{-0.0005889x}[1 - 0.0005889x]$$
$$= 0$$

Since $R' = 0$ when $x = 1/0.0005889 \approx 1698$ units, we have $p = 81.0915e^{-0.0005889(1698)} \approx \$29.83$.

46.

$$A = Ve^{-0.04t}$$
$$= 100{,}000e^{0.75\sqrt{t}}e^{-0.04t}$$
$$= 100{,}000e^{(0.75\sqrt{t} - 0.04t)}$$
$$A'(t) = 100{,}000\left(\frac{0.75}{2\sqrt{t}} - 0.04\right)e^{(0.75\sqrt{t} - 0.04t)} = 0$$
$$\frac{0.75}{2\sqrt{t}} = 0.04$$
$$\sqrt{t} = \frac{0.75}{(0.04)(2)} = 9.375$$
$$t = 87.89 \approx 88$$

The timber should be harvested in 2078 to maximize the present value.

48. (a) If $I_0 = 1$, then we have

$$R = \frac{\ln I}{\ln 10} \text{ and } 8.3 = \frac{\ln I}{\ln 10}.$$

Therefore, $I = e^{8.3 \ln 10} \approx 199{,}526{,}231.5$.

(b) $2R = \frac{\ln I}{\ln 10}$ implies that $I = e^{2R \ln 10} = (e^{R \ln 10})^2$.

The intensity is squared if R is doubled.

(c) $\frac{dR}{dI} = \frac{1}{\ln 10}\left(\frac{1}{I}\right) = \frac{1}{I \ln 10}$

Review Exercises for Chapter 4

2. $16^{3/2} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

4. $\left(\frac{27}{8}\right)^{-1/3} = \left(\frac{8}{27}\right)^{1/3} = \frac{2}{3}$

6. $(9^{1/3})(3^{1/3}) = (9 \cdot 3)^{1/3} = 27^{1/3} = 3$

8. $\frac{1}{4}\left(\frac{1}{2}\right)^{-3} = \frac{1}{4}(2)^3 = \frac{8}{4} = 2$

10.

$$32^{-x} = 2^{3x+1}$$
$$(2^5)^{-x} = 2^{3x+1}$$
$$-5x = 3x + 1$$
$$8x = -1$$
$$x = -\frac{1}{8}$$

12.

$$x^{5/2} = 243 = 3^5$$
$$\sqrt{x} = 3$$
$$x = 9$$

14.

$$e^{-5} = e^{2x+1}$$
$$-5 = 2x + 1$$
$$x = -3$$

16.

$$4x^2 = e^5$$
$$x^2 = \frac{1}{4}e^5$$
$$x = \pm\frac{e^{5/2}}{2}$$

18. $P = 36.182(1.193)^t$

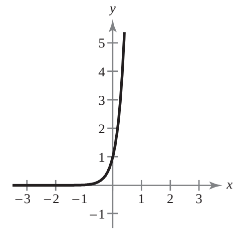
(a) 1999: $P(9) = 36.182(1.193)^9 \approx \$177.1$ million

2001: $P(11) \approx \$252.1$ million

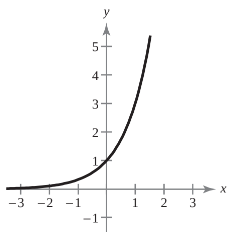
2003: $P(13) \approx \$358.8$ million

(b) Answers will vary.

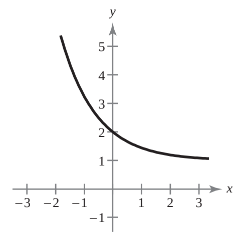
20. $g(x) = 16^{3x/2}$



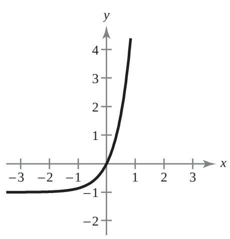
22. $g(t) = \left(\frac{1}{3}\right)^{-t} = 3^t$



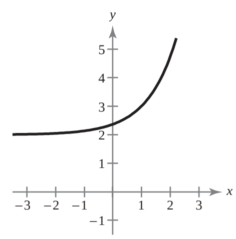
24. $g(x) = \left(\frac{2}{3}\right)^{2x} + 1$



26. $g(x) = e^{2x} - 1$



28. $g(x) = 2 + e^{x-1}$



30. $y = 1096e^{-0.39t}$

If $t = 20$, $y = 0.449 < 1$, which indicates that the species is endangered.

32. $f(t) = e^{4t} - 1$

(a) $f(0) = e^0 - 1 = 0$

(b) $f(2) = e^{8t} - 1$

(c) $f\left(-\frac{3}{4}\right) = e^{-3} - 1$

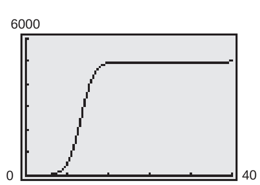
34. $g(x) = \dfrac{24}{1 + e^{-0.3x}}$

(a) $g(0) = \dfrac{24}{1 + 1} = 12$

(b) $g(300) \approx 24$

(c) $g(1000) \approx 24$

36. (a) $P = \dfrac{5000}{1 + 4999e^{-0.8t}}, \quad 0 \le t$



(b) When $t = 5$, $P = 54$ students.

(c) Yes, as $t \to \infty$, $P \to 5000$.

38.

n	1	2	4	12	365	Continuous
A	16,035.68	16,310.19	16,453.31	16,551.02	16,598.95	16,600.58

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 5000\left(1 + \frac{0.06}{n}\right)^{20n}$$

$$A = Pe^{rt} = 5000e^{0.06(20)} \approx 16{,}600.58$$

(Continuous)

40. (a) $A = P\left(1 + \frac{r}{n}\right)^{nt} = 2000\left(1 + \frac{0.065}{12}\right)^{12(10)} \approx \$3824.37$

(b) $A = Pe^{rt} = 2000e^{0.0625(10)} \approx \$3831.08$

Account (b) will be greater.

42. (a) $r_{eff} = \left(1 + \frac{r}{n}\right)^{n} - 1 = \left(1 + \frac{0.0825}{4}\right)^{4} - 1 \approx 0.0851$ or 8.51%

(b) $r_{eff} = \left(1 + \frac{0.0825}{12}\right)^{12} - 1 \approx 0.0857$ or 8.57%

44. $P = \dfrac{20{,}000}{\left(1 + \frac{0.08}{12}\right)^{12(5)}} \approx \$13{,}424.21$

46. 1996: $R(6) \approx \$362.6$ million

2000: $R(10) \approx \$930.9$ million

2003: $R(13) \approx \$918.0$ million

48. $y = 4e^{\sqrt{x}}$

$$y' = 4e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right) = \frac{2e^{\sqrt{x}}}{\sqrt{x}}$$

50. $y = x^2e^x$

$$y' = x^2e^x + 2xe^x = (x^2 + 2x)e^x$$

52. $y = (2e^{3x})^{1/3}$

$$y' = \frac{1}{3}(2e^{3x})^{-2/3}(6e^{3x}) = \frac{2e^{3x}}{(2e^{3x})^{2/3}} = 2^{1/3}e^x$$

54. $y = 10(1 - 2e^x)^{-1}$

$$y' = -10(1 - 2e^x)^{-2}(-2e^x) = \frac{20e^x}{(1 - 2e^x)^2}$$

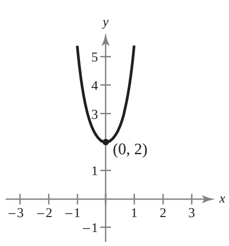
56. $f(x) = 2e^{x^2}$

$f'(x) = 4xe^{x^2}$

$f''(x) = 4x(2xe^{x^2}) + 4e^{x^2} = (8x^2 + 4)e^{x^2}$

(0, 2) is a relative minimum.

No inflection points nor asymptotes



58. $f(x) = \dfrac{e^x}{x^2}$

$$f'(x) = \frac{x^2e^x - 2xe^x}{x^4} = \frac{e^x(x - 2)}{x^3}$$

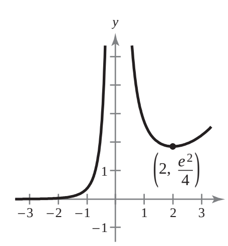
$$f''(x) = \frac{e^x(x^2 - 4x + 6)}{x^4}$$

Relative minimum: $\left(2, \frac{e^2}{4}\right)$

Horizontal asymptote: $y = 0$

No points of inflection

Vertical asymptote: $x = 0$



60. $f(x) = \dfrac{x^2}{e^x}$

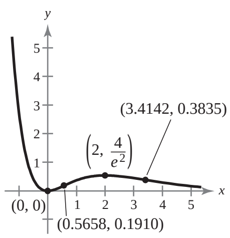
$f'(x) = \dfrac{x(2-x)}{e^x}$

$f''(x) = \dfrac{x^2 - 4x + 2}{e^x}$

$(0, 0)$ is a relative minimum; $\left(2, \dfrac{4}{e^2}\right)$ is a relative maximum.

Points of inflection: $(3.4142, 0.3835)$, $(0.5658, 0.1910)$

Horizontal asymptote: $y = 0$



62. $f(x) = xe^{-2x}$

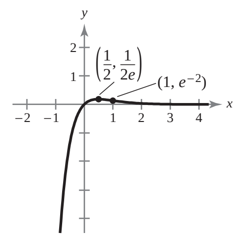
$f'(x) = (1 - 2x)e^{-2x}$

$f''(x) = 4(x - 1)e^{-2x}$

Relative maximum: $\left(\dfrac{1}{2}, \dfrac{1}{2e}\right)$

Point of inflection: $(1, e^{-2})$

Horizontal asymptote: $y = 0$



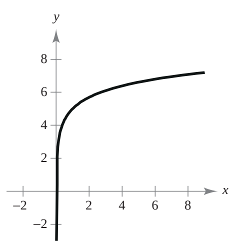
64. $\ln 0.6 \approx -0.5108$

$e^{-0.5108} \approx 0.6$

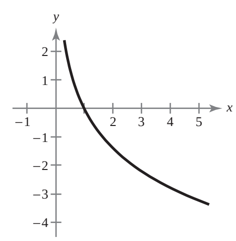
66. $e^{-4} \approx 0.0183$

$\ln 0.0183 \approx -4$

68. $y = 5 + \ln x$



70. $y = -2 \ln x$



72. $\ln \sqrt[3]{x^2 - 1} = \dfrac{1}{3}\ln[(x-1)(x+1)] = \dfrac{1}{3}\ln(x-1) + \dfrac{1}{3}\ln(x+1)$

74. $\ln\dfrac{x^2}{x^2+1} = 2\ln x - \ln(x^2+1)$

76. $\ln\left(\dfrac{x-1}{x+1}\right)^2 = 2\ln(x-1) - 2\ln(x+1)$

78. $e^{\ln(x+2)} = x + 2 = 5 \Rightarrow x = 3$

80. $\ln x = 2e^5 \Longrightarrow x = e^{2e^5}$

82. $\ln x - \ln(x+1) = \ln\left(\frac{x}{x+1}\right) = 2$

$$\frac{x}{x+1} = e^2$$

$$x = xe^2 + e^2$$

$$x(1 - e^2) = e^2$$

$$x = \frac{e^2}{1 - e^2}$$

Sine $x < 0$, no solution.

84. $4e^{2x-3} = 5$

$$e^{2x-3} = \frac{5}{4}$$

$$2x - 3 = \ln\left(\frac{5}{4}\right)$$

$$x = \frac{1}{2}\left[3 + \ln\left(\frac{5}{4}\right)\right]$$

86. $2 \ln x + \ln(x-2) = 0$

$$\ln[x^2(x-2)] = 0$$

$$x^2(x-2) = 1$$

$$x^3 - 2x^2 - 1 = 0$$

$$x \approx 2.2056$$

88. $e^{-0.01x} = 5.25$

$$-0.01x = \ln[5.25]$$

$$x = -100 \ln(5.25) \approx -165.8228$$

90. $500(1.075)^{120x} = 100{,}000$

$$(1.075)^{120x} = 200$$

$$120x \ln(1.075) = \ln 200$$

$$x = \frac{\ln 200}{120 \ln(1.075)} \approx 0.6105$$

92. $\frac{50}{1 - 2e^{-0.001x}} = 1000$

$$1 - 2e^{-0.001x} = \frac{1}{20}$$

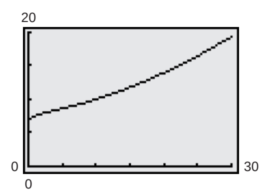
$$2e^{-0.001x} = \frac{19}{20}$$

$$e^{-0.001x} = \frac{19}{40}$$

$$-0.001x = \ln\left(\frac{19}{40}\right)$$

$$x = 1000 \ln\left(\frac{40}{19}\right) \approx 744.4405$$

94. (a)



(b) $\omega = 12$ when $t \approx 15.5$ or, 1995.

(c) Answers will vary.

96. $y = \ln\sqrt{x} = \frac{1}{2}\ln x$

$$y' = \frac{1}{2x}$$

98. $y = \ln\frac{x^2}{x+1} = 2\ln x - \ln(x+1)$

$$y' = \frac{2}{x} - \frac{1}{x+1} = \frac{x+2}{x(x+1)}$$

100. $f(x) = \ln e^{x^2} = x^2$

$$f'(x) = 2x$$

102. $y = \frac{x^2}{\ln x}$

$$y' = \frac{2x \ln x - x}{(\ln x)^2}$$

104. $y = \ln \sqrt[3]{x^3 + 1} = \frac{1}{3}\ln(x^3 + 1)$

$y' = \frac{1}{3(x^3 + 1)}(3x^2) = \frac{x^2}{x^3 + 1}$

106. $f(x) = \ln \frac{x}{\sqrt{x + 1}} = \ln x - \frac{1}{2}\ln(x + 1)$

$f'(x) = \frac{1}{x} - \frac{1}{2(x + 1)}$

108. $y = \ln\left[e^{2x}\sqrt{e^{2x} - 1}\right] = 2x + \frac{1}{2}\ln(e^{2x} - 1)$

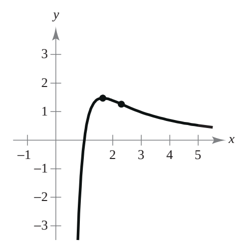
$y' = 2 + \frac{e^{2x}}{e^{2x} - 1} = \frac{3e^{2x} - 2}{e^{2x} - 1}$

110. $y = \frac{8 \ln x}{x^2}$

$y' = 8\left(\frac{1 - 2\ln x}{x^3}\right)$

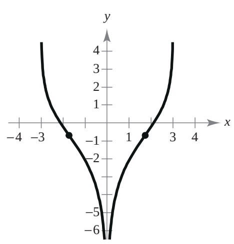
$y'' = 8\left(\frac{6\ln x - 5}{x^4}\right)$

$(e^{1/2}, 1.472)$ is a relative maximum.
$(e^{5/6}, 1.259)$ is a point of inflection.



112. $y = \ln \frac{x^2}{9 - x^2} = 2\ln x - \ln(9 - x^2)$

No relative extrema
Inflection points: $\left(\sqrt{3}, -0.693\right), \left(-\sqrt{3}, -0.693\right)$



114. $\log_2 32 = \log_2 2^5 = 5\log_2 2 = 5$

116. $\log_4 \frac{1}{64} = \log_4 4^{-3} = -3\log_4 4 = -3$

118. $\log_4 12 = \frac{\ln 12}{\ln 4} \approx 1.7925$

120. $\log_4 125 = \frac{\ln 125}{\ln 4} \approx 3.483$

122. $y = \log_{10} \frac{3}{x} = \log_{10} 3 - \log_{10} x$

$y' = \frac{-1}{\ln 10} \cdot \frac{1}{x} = \frac{-1}{x \ln 10}$

124. $y = \log_{16}(x^2 - 3x)$

$y' = \frac{1}{\ln 16} \cdot \frac{2x - 3}{x^2 - 3x} = \frac{2x - 3}{(x^2 - 3x)\ln 16}$

126. $C = P(1.05)^t$

(a) $C = 24.95(1.05)^{10} \approx \$40.64$

(b) $C'(t) = P\ln(1.05)(1.05)^t$

$= P\ln(1.05)(1.05) \approx 0.0512P$

128. $P = P_0 e^{0.025t}$, $P_0 =$ initial population

(a) $2P_0 = P_0 e^{0.025t}$

$2 = e^{0.025t}$

$\ln 2 = 0.025t$

$t = \dfrac{\ln 2}{0.025} \approx 27.7$ years

(b) $3P_0 = P_0 e^{0.025t}$

$3 = e^{0.025t}$

$\ln 3 = 0.025t$

$t = \dfrac{\ln 3}{0.025} \approx 43.9$ years

130. $\dfrac{1}{2} = e^{k(5.2)}$

$k = \dfrac{\ln(1/2)}{5.2} \approx -0.1333$

$0.1 = 0.5e^{-0.1333t}$

$0.2 = e^{-0.1333t}$

$t = \dfrac{\ln(0.2)}{-0.1333} \approx 12.1$ years

132. Using the data points $(4, 266.6)$ and $(13, 2338.1)$, you obtain the model

$P = 101.566(1.2729)^t = 101.566e^{0.2413t}$

For 2006, $P(16) \approx \$4825$ million.