MA 22000 Lesson 13 Notes, Section 1.5 (2^{nd} half of text)
Limits

I encourage all students to read the first paragraph of this lesson from the text (page 49). It will help you understand the concept of a 'limit'. This paragraph says as a weight approaches 10 pounds, the length of the stretch of the spring will approach a certain number called the 'limit'. Let us suppose that limit is 8 inches. (If any more weight than 10 points is put on the spring, it will break.) We could say 'the limit of the length of the stretched spring as the weight approaches 10 pounds is 8 inches. This would be written as $\lim_{w \to 10} (\text{spring}) = 8$ (the limit of the length of the spring as weight *w* approaches 10 pounds is 8 inches). The general limit notation is $\lim_{x \to c} f(x) = L$, which is read ' the limit of $f(x)$ as *x* approaches *c* is *L*.

There are many different strategies used in calculus to find limits. One approach is to evaluate the function for numbers very close to *c*, slightly larger and/or slightly smaller. Examine these examples.

Ex 1:
$\lim_{x \to 4}(2x+3)$ where $f(x) = 2x+3$ Select values of *x* slightly smaller or slightly larger than 4.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	10.8	10.98	10.998	11.002	11.02	11.2

You can examine that the closer the *x* value is to 4, the function value is closer to 11. We say $\lim_{x \to 4}(2x+3) = 11$.
Notice say a 'direct substitution' of 4 into the function value $2x + 3$ yielded the limit value of 11.

Ex 2:
$\lim_{x \to 3}\left(\frac{x^2-9}{x-3}\right)$ where $g(x) = \frac{x^2-9}{x-3}$ Select values of *x*, slightly smaller or slightly larger than 3.

x	2.9	2.99	2.999	3.001	3.01	3.1
$g(x)$	5.9	5.99	5.999	6.001	6.01	6.1

You can examine that the close the *x* value is to 3, the function value is closer to 6. We say

$$\lim_{x \to 3}\left(\frac{x^2-9}{x-3}\right) = 6$$

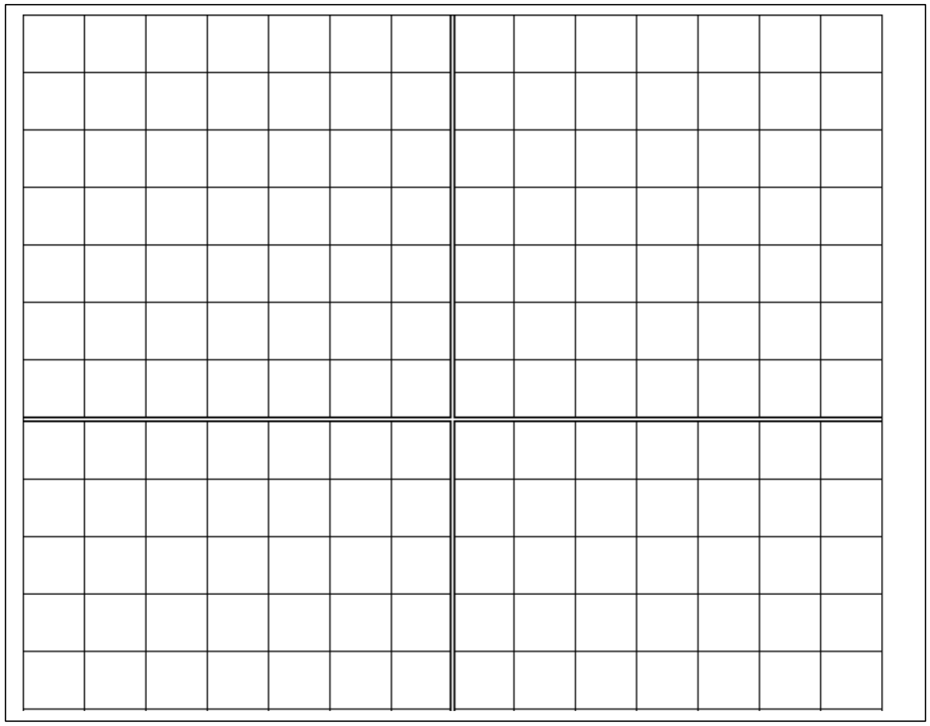
Notice that this time a 'direct substitution' would not work, because a zero denominator would result.

Sometimes a 'direct substitution' works, sometimes it does not. You can use tables such as above to determine limits sometimes. Sometimes looking at a graph helps. Look at figure 1.52 on page 49 of the text. This graph corresponds to the limit, $\lim_{x \to 1}(x^2+1)$. The red arrows show that approaching the *x* value 1 **from either the left (values smaller than 1) or from the right**

(values larger than 1), yield a function value (y value) of 2. We can easily view that $\lim_{x \to 1}(x^2 + 1) = 2$.

Ex 2:   Graph $f(x) = \dfrac{|x-2|}{x-2}$.

As you follow the graph toward $x = 2$ from the left or from the right, does it appear to be approaching a certain y value?



Let's also make a table.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	-1	-1	-1	1	1	1

Using the table does the $\lim_{x \to 2}\left(f(x)\right)$ appear to be approaching the same function value from 'the left' and from 'the right'? We say $\lim_{x \to 2}\left(f(x)\right)$ does not exist.

After examination of the example above and the other examples shown on page 50, the following are three important conclusions.

1) Saying that the limit of $f(x)$ as x approaches c is L means that the function value gets very, very close to L as x gets closer and closer to c.
2) For a limit L to exist, you must allow x to approach c from **either side of *c***. For the limit to exist, the value found by approaching from either the left or the right must be the same.
3) The function does not have to be defined at c in order to have a limit as $x \to c$. In other words, a limit may exist even though the function value does not.

Techniques for Evaluating Limits:

1. With a polynomial function (or many other functions), direct substitution will work. See the examples below.

   $a)\quad \lim_{x\to(-2)}(x^2-x)=(-2)^2-(-2)=4+2=6$

   $b)\quad \lim_{n\to3}(2n-4)=2(3)-4=6-4=2$

   $c)\quad \lim_{a\to5}\sqrt{2a+6}=\sqrt{2(5)+6}=\sqrt{16}=4$

   $d)\quad \lim_{n\to2}\left(\frac{2n-5}{n+1}\right)=\frac{2(2)-5}{2+1}=\frac{-1}{3}\text{ or }-\frac{1}{3}$

2. Sometimes with a rational function or rational expression, you can write an equivalent expression for the function by simplifying, then use 'direct substitution' by using the replacement theorem (see page 53 of text).

   $a)\quad \lim_{x\to3}\left(\frac{x^2-9}{x-3}\right)=\lim_{x\to3}\left(\frac{(x+3)(x-3)}{x-3}\right)=\lim_{x\to3}(x+3)=3+3=6$

   $b)\quad \lim_{c\to5}\left(\frac{c^2-4c-5}{c-5}\right)=\lim_{c\to5}\left(\frac{(c-5)(c+1)}{c-5}\right)=\lim_{c\to5}(c+1)=5+1=6$

   $c)\quad \lim_{\Delta x\to0}\left(\frac{3(x+\Delta x)-2-(3x-2)}{\Delta x}\right)=\lim_{\Delta x\to0}\left(\frac{3x+3(\Delta x)-2-3x+2}{\Delta x}\right)$

   $=\lim_{\Delta x\to0}\left(\frac{3(\Delta x)}{\Delta x}\right)=\lim_{\Delta x\to0}3=3$

Ex 3: Find each limit.

$a)\quad \lim_{x\to5}(2x+3)$

$b)\quad \lim_{a\to(-3)}(2a^2-5a+7)=$

$c)\quad \lim_{x\to4}\sqrt{21+x}$

$d)\quad \lim_{b\to2}\sqrt{x-5}$

$e)\quad \lim_{m\to2}\left(\frac{\frac{1}{x}-\frac{1}{x+1}}{x}\right)$

Ex 4: Find each limit, if it exists.

$a)\quad \lim_{x\to -5}\left(\frac{x^2-25}{x+5}\right)$

$b)\quad \lim_{x\to 4}\left(\frac{x-4}{x^2-8x+16}\right)$

$c)\quad \lim_{t\to 2}\left(\frac{t^2+3t-10}{t^2-4}\right)$

$d)\quad \lim_{\Delta x\to 0}\left(\frac{4(x+\Delta x)+3-(4x+3)}{\Delta x}\right)$

$e)\quad \lim_{\Delta r\to 0}\left(\frac{(r+\Delta r)^2-2(r+\Delta r)-1-(r^2-2r-1)}{\Delta r}\right)$