MA 222: Summary of Differential Equations Lessons

Hoffman

Part I
Introduction

A **differential equation** is an equation involving differentials/derivatives. The answer to a differential equation is a function (usually written as $y = \dots$) whose differential/derivatives satisfy the equation. As a simple example, consider $\frac{dy}{dx} = 1$. The answer to this differential equation is a function y such that the derivative of y is 1. Without writing anything down, you can confirm that a possible answer is $y = x$. However, another answer is $y = x+1$ because the derivative of $x+1$ is also 1. In fact, any function $y = x+c$, where c is any constant, satisfies the differential equation. We call $y = x + c$ the **general solution** since it is the general form of the solutions. A **particular solution** has a concrete c value. If the problem told us that we needed $y = 3$ when $x = 1$, then we would have need $3 = 1 + c$, that is, $c = 2$. The particular solution would then be $y = x + 2$.

Part II
Finding General Solutions

1 Classification

The first thing to consider when given a differential equation is the type of derivatives. The **order** of a differential equation is the order of the highest derivative. If you only have first derivatives or differentials $(dx, dy, \frac{dy}{dx}, y')$, then you have a first order differential equation. If you have a second derivative $(\frac{d^2y}{dx^2})$, then you have a second order differential equation. Note that for convenience, the notation D is used for derivatives where $D = \frac{d}{dx}$. Hence, $D^2y = \frac{d^2y}{dx^2}$ is another form in which you may encounter second derivatives. If you have a first order equation, go to Section 2. If you have a second order, go to Section 3.

2 First Order Differential Equations

There are basically two methods to solve a first order equation: **Separation of Variables** and the **Integrating Factor Method**. Almost always, if you can separate the variables, then the first method will be easier. The Integrating Factor Method will always work, but can get messy. To determine if you can separate variables, see if you can multiply/divide factors to get all of the "x things" with dx and the "y things" with dy. You will want your equation to look like

$$M(x)dx + N(y)dy = 0.$$

where $M(x)$ is a function of x such as $x^3, \cos(x), \frac{1}{x \sin x}, \ln(x)$, etc., and $N(y)$ is a function of y. There are some functions that cannot be put into this form. For instance, if you have a term that is a sum of an "x thing" and a "y thing", you will not be able to separate the variables. For this mixed factor cannot go with

dx as it contains a y, nor can it go with dy since it contains something with x. If you can separate the variables, go to Section 2.1. If you cannot separate the variables, go to Section 2.2.

2.1 Separation of Variables

The separation of variables method works when we can write our equation as

$$M(x)dx + N(y)dy = 0.$$

To finish the problem, simply integrate both sides. You get

$$\begin{aligned}\int (M(x)dx + N(y)dy) &= \int M(x)dx + \int N(y)dy \\ &= \int 0 \\ &= 0.\end{aligned}$$

When you calculate the integrals $\int M(x)dx$ and $\int N(y)dy$, you will get arbitrary constants "$+C$" as they are indefinite integrals. Generally, we collect the constants on the right-hand-side as a single constant c. Therefore, if $m(x)$ is the antiderivative of $M(x)$ and $n(y)$ the antiderivative of y, then the final answer you write as $m(x) + n(y) = c$.

2.2 Integrating Factor Method

If you cannot separate the variables, then write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Next, calculate the integrating factor, IF. IF$= e^{\int P(x)dx}$. Third, multiply both sides of the standard form equation by the integrating factor. On the left-hand-side, you have

$$\text{IF}\left(\frac{dy}{dx} + P(x)y\right).$$

Because of the way we chose the integrating factor, the following always holds:

$$\text{IF}\left(\frac{dy}{dx} + P(x)y\right) = \frac{d}{dx}\left(\text{IF} \cdot y\right).$$

Thus, we have

$$\frac{d}{dx}\left(\text{IF} \cdot y\right) = \text{IF} \cdot Q(x).$$

Next, integrate both sides of the equation. The integral cancels with the derivative on the left so we get

$$\text{IF} \cdot y = \int \text{IF} \cdot Q(x)dx.$$

After you perform the integration on the right, you will have a "$+c$." The last step is to divide by the integrating factor to put your answer in the form $y = \ldots$

On the formula sheet, there is a summary of this method which says $y' + P(x)y = Q(x)$ has solution y where

$$ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + C.$$

The first equation just shows you what the standard form is so you can identify $P(x)$ and $Q(x)$. The second formula combines all of the steps above into one equation so you can just plug in $P(x)$, $Q(x)$, evaluate the necessary integrals and then skip to the last step of getting y by itself.

3 Second Order Equations

In general, our second order equation will have have the form

$$(b_2D^2 + b_1D + b_0)y = f(x), \text{ or equivalently } b_2\frac{d^2y}{dx^2} + b_1\frac{dy}{dx} + b_0y = f(x).$$

Here, b_2, b_1, b_0 are just constants. If $f(x) = 0$ for all x, i.e., the right-hand-side is just zero, then we say that the equation is **homogeneous**. If $f(x) \neq 0$, then the equation is **nonhomogeneous**. Skip to Section 3.1 for the former case and Section 3.2 for the latter.

3.1 Homogeneous Second Order

Our equation looks like

$$(b_2D^2 + b_1D + b_0)y = 0 \text{ or } b_2\frac{d^2y}{dx} + b_1\frac{dy}{dx} + b_0y = 0.$$

We take this equation and write the **auxiliary equation**: $b_2m^2 + b_1m + b_0 = 0$. Since b_2, b_1, and b_0 are just constants, this is a quadratic equation. Find the roots of the quadratic by whatever means; factoring is the quickest, but sometimes you will have to resort to the quadratic formula. A quadratic always has two roots. If we label them m_1, m_2, then there are three cases: If m_1, m_2 are real (that is, not imaginary) and distinct ($m_1 \neq m_2$), then go to Section 3.1.1. If $m_1 = m_2$, go to Section 3.1.2. If m_1, m_2 are complex conjugates, that is, you can write them as $a \pm bi$ for some a, b, then go to Section 3.1.3.

3.1.1 Real Distinct Roots

If your roots are real and distinct, then the general solution is

$$y = c_1e^{m_1x} + c_2e^{m_2x}.$$

3.1.2 Repeated Roots

If $m_1 = m_2$, then the solution has the form

$$y = c_1e^{m_1x} + c_2xe^{m_1x}.$$

Note the x inbetween the c_2 and the exponential. Also, it doesn't matter whether you use m_1 or m_2 because the roots are equal.

3.1.3 Complex Roots

If the roots are complex, then we think of them as $a \pm bi$ and the general solution has the form

$$y = e^{ax}\left(c_1\cos(bx) + c_2\sin(bx)\right).$$

Note that c_1 and c_2 may be complex numbers, but rarely are.

3.2 Nonhomogeneous Second Order

Our equation has the form

$$(b_2D^2 + b_1D + b_0)y = f(x), \text{ or } b_2\frac{d^2y}{dx^2} + b_1\frac{dy}{dx} + b_0y = f(x).$$

There are two main steps to solving these types of equations. The first is to find what is called the complimentary solution. We usually denote this y_c. The **complimentary solution** is just the solution to the homogeneous equation. In other words, pretend that $f(x) = 0$ for the time being and solve the equation according to the method laid out in Section 3.1. The second step is find a **particular solution** to the nonhomgeneous equation. To do this, we condsider $f(x)$. Calculate the first and second derivatives of $f(x)$. Then, see what types of functions you get, whether there are linear terms, quadratic terms, exponential terms, trig terms, or some product of the preceding, or whatever. Ignore the constants out front, just look for the form of the terms. Then we guess that a particular solution will be some combination of these terms. We write $y_p = A \cdot T_1 + B \cdot T_2 + \cdots$ where T_1, T_2 are the terms from $f(x), f'(x)$, and $f''(x)$ and $A, B, C, D, \ldots$ are arbitrary constants. You should be able to express $f(x), f'(x)$, and $f''(x)$ as y_p for some choice of constants. To find exactly what the constants should be, we plug in our guess for y_p into the differential equation. In other words, write down $(b_2D^2 + b_1D + b_0)y_p$. We compare the result to $f(x)$, specifically the coeffcients. We need constant coefficients such that $(b_2D^2 + b_1D + b_0)y_p = f(x)$. Think in terms of finding constants to match the left to the right. Once you find the right constants, you will have found y_p. The final step is that the general solution is the sum of the complimentary solution and the particular solution: $y = y_c + y_p$.

Part III
Finding Particular Solutions

No matter what type of equation you have, the process for finding a particular solution is the same. Use Section II to find the general solution, then use the extra pieces of information given to you to find what the constants in the general solution need to be. The problem may say something like $y = 1$ and $Dy = 0$ when $x = 0$. In this case, take your general solution, plug in zero, then set the result equal to 1. Next, you would calculate the derivative of the general solution, plug in $x = 0$ to that, then set the result equal to zero. The two equations you have would allow you to find two unknown constants c_1 and c_2. In general, a first order equation will have one unknown constant, so if you are asked to find a particular solution, you will be given one extra piece of information. A second order general soltuion will have two unknown constants, so if you are asked to find a particular solution, you will be given two extra pieces of information. Remember though that you always find the general solution first, without giving any thought to the conditions given. Only after you have obtained the general solution do you attempt to use conditions to find the particular solution.

Part IV
Applications

4 Exponential Growth and Decay

Many quantities change in proportion to their size. Examples would include bacteria growth, radioactive decay, and zombie infections. In such cases, the equation

$$N = N_0 e^{kt}$$

models the quantity with respect to time where N is the amount present at time t, N_0 is the initial amount at $t = 0$ and k is called the constant of proportionality. Exponential growth corresponds to $k > 0$ whereas exponential decay corresponds to $k < 0$.

5 Newton's Law of Cooling

If we place a body in a medium of constant temperature, how the body's temperature changes is modeled by

$$T = T_m + (T_0 - T_m)e^{-kt},$$

where T is the temperature of the body at time t, T_0 is the temperature of the body at $t = 0$, T_m is the (constant) temperature of the medium, and $k > 0$. You should memorize this formula.

6 Harmonic Motion

The only application for our second order methods that we learned involve harmonic motion. Specifically, we have the motion of a weight hung on a spring which is attached to some unmovable block. All distances are measured in feet, forces in pound/feet, and masses in terms of slugs (1 slug is 32 lbs). Let x be the displacement variable with $x = 0$ the equilibrium of the weight on the spring. Let positive x indicate a stretching of the spring downward and negative x indicate a compression of the spring upward. As usual, t will represent time. Then the motion of the weight is given by the second order equation

$$m\frac{d^2x}{dt} + b\frac{dx}{dt} + kx = f(t).$$

Here, m is the mass of the weight, b models the damping effect, k is the Hooke's constant for the given spring and $f(t)$ is an external force on the weight (as in someone hitting the weight with their hand). You find m by converting the weight's weight to a mass (in slugs). You will be given b with a statement such as, "The damping force is equal to twice the velocity." This would indicate $b = 2$. You find k based on how a given weight stretches the spring using Hooke's Law $F = kx$. Plug in the weight of the weight for F and how far the spring stretches for x, solve for k. The external force (if there is any) will simply be given as a function $f(t) = \ldots$ After you get all of these values collected, approach the equation using the methods of Section 3.