Hoffman

1. (7 points) Differentiate $g(x) = \dfrac{\sin(\sqrt{x})}{\ln(4x)}$.

Solution: We use the quotient rule:

$$\begin{aligned} g'(x) &= \frac{\ln(4x)\frac{d}{dx}(\sin(\sqrt{x})) - \sin(\sqrt{x})\frac{d}{dx}(\ln(4x))}{(\ln(4x))^2} \\ &= \frac{\ln(4x)(\cos(\sqrt{x})\frac{d}{dx}(\sqrt{x})) - \sin(\sqrt{x})(\frac{1}{4x}\cdot\frac{d}{dx}(4x))}{(\ln(4x))^2} \\ &= \frac{\ln(4x)\cos(\sqrt{x})\frac{1}{2\sqrt{x}} - \sin(\sqrt{x})(\frac{1}{4x}\cdot 4)}{(\ln(4x))^2} \\ &= \frac{\frac{\ln(4x)\cos(\sqrt{x})}{2\sqrt{x}} - \frac{\sin(\sqrt{x})}{x}}{(\ln(4x))^2} \end{aligned}$$

2. (7 points) Differentiate $f(x) = x^3 e^{\sin(\ln(x))}$.

Solution: We use the product rule and several chain rule applications:

$$\begin{aligned} f'(x) &= x^3\left(\frac{d}{dx}(e^{\sin(\ln(x))})\right) + e^{\sin(\ln(x))}\frac{d}{dx}(x^3) \\ &= x^3 e^{\sin(\ln(x))}\frac{d}{dx}(\sin(\ln(x))) + e^{\sin(\ln(x))}3x^2 \\ &= x^3 e^{\sin(\ln(x))}\cos(\ln(x))\frac{d}{dx}(\ln(x)) + e^{\sin(\ln(x))}3x^2 \\ &= x^3 e^{\sin(\ln(x))}\cos(\ln(x))\frac{1}{x} + e^{\sin(\ln(x))}3x^2 \\ &= x^2 e^{\sin(\ln(x))}\cos(\ln(x)) + 3x^2 e^{\sin(\ln(x))}. \end{aligned}$$

3. (9 points) Differentiate $f(x) = \ln\left(\frac{x^4}{e^x \cos^2(x)}\right)$. You *must* use the Rules of Logarithms discussed in class *and* simplify completely.

Solution: Using the Rules of Logarithms:

$$\begin{aligned} f(x) &= \ln\left(\frac{x^4}{e^x \cos^2(x)}\right) \\ &= \ln(x^4) - \ln(e^x \cos^2(x)) \\ &= \ln(x^4) - \big(\ln(e^x) + \ln(\cos^2(x))\big) \\ &= 4\ln(x) - x\ln(e) - 2\ln(cos(x)) \\ &= 4\ln(x) - x - 2\ln(\cos(x)). \end{aligned}$$

Now, we differentiate:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(4\ln(x) - x - 2\ln(\cos(x))\right) \\ &= \frac{d}{dx}(4\ln(x)) - \frac{d}{dx}(x) - \frac{d}{dx}(2\ln(\cos(x))) \\ &= \frac{4}{\ln x} - 1 - \frac{2}{\cos(x)}\frac{d}{dx}(\cos(x)) \\ &= \frac{4}{\ln x} - 1 + \frac{2\sin(x)}{\cos(x)} \\ &= \frac{4}{\ln x} - 1 + 2\tan(x). \end{aligned}$$

4. (7 points) The number of bacteria N in some particular culture after t minutes is modeled by $N = 20e^{.5t}$. Find the rate of change to one decimal after 5 minutes.

Solution: Finding a rate of change involves finding the derivative. $\frac{d}{dt}(N) = 20e^{.5t} \cdot (.5) = 10e^{.5t}$. This formula gives the rate of change at any time t. We are asked about $t = 5$, so we plug $10e^{.5 \cdot 5} = 10e^{2.5}$ into our calculator to get 121.8.

5. (8 points) Differentiate $y = x^{\sin 2x}$.

Solution: First, we take the natural log of both sides: $\ln(y) = \ln(x^{\sin 2x})$. Using the Power Rule for Logarithms, $\ln(x^{\sin 2x}) = \sin(2x)\ln(x)$. Now we hit both sides with $\frac{d}{dx}$, remembering that y is a function of x, so we have to use a chain rule:

$$\begin{aligned}
\frac{d}{dx}\ln(y) &= \frac{d}{dx}(\sin(2x)\ln(x)) \\
\frac{1}{y}\frac{d}{dx}(y) &= \sin(2x)\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(\sin 2x) \\
\frac{1}{y}\frac{dy}{dx} &= \sin(2x)\cdot\frac{1}{x} + \ln x\cos(2x)\frac{d}{dx}(2x) \\
\frac{1}{y}\cdot y' &= \frac{\sin(2x)}{x} + \ln(x)2\cos(2x) \\
y' &= y\left[\frac{\sin(2x)}{x} + \ln(x)2\cos(2x)\right] \\
y' &= x^{\sin 2x}\left[\frac{\sin(2x)}{x} + \ln(x)2\cos(2x)\right].
\end{aligned}$$

The last line used the orignal statement of the problem that $y = x^{\sin 2x}$.

6. (9 points) Find the minimum *and* maximum of the function $f(x) = \pi x e^{-x}$, *or* write NONE where applicable. State both where and what the extreme values are, that is, the x and $f(x)$ values.

Solution: We can only have a min/max if the derivative is zero, so we differentiate using the product rule: $f'(x) = \pi x(-e^{-x}) + e^{-x}(\pi) = \pi e^{-x}(1-x)$. Setting this equal to zero, we notice the first factor is never zero (because exponential functions are always positive). The second factor is zero precisely for $x = 1$. Thus, our only candidate for a min/max is $x = 1$. To see which it might be, we check the sign (positive or negative) of the first derivative to the left and right. To that end, $f'(0) = \pi e^0(1-0) = \pi > 0$ and $f'(2) = \frac{\pi}{e^2}\cdot -1 < 0$. Since the first derivative goes from positive to negative, the function goes from increasing to decreasing, so $x = 1$ is a maximum. We plug it in to find the actual value $f(1) = \pi e^{-1}\cdot 1$. Therefore, the final answers are Maximum: $\left(1, \frac{\pi}{e}\right)$ and Minimum: NONE.

7. (8 points) Find the area between the the y-axis, the x-axis, the curve $y = (x+1)^3$ and the line $x = 1$. Express your answer as a fraction.

Solution: Area under a curve (assuming the curve lies above the x-axis) is given by an integral. The descrption of the problem means we have to solve $\int_0^1 (x+1)^3 dx$. Let $u = x + 1$. Then $du = dx$, so we have

$$\begin{aligned} \int_0^1 (x+1)^3 dx &= \int_1^2 u^3 du \\ &= \left.\frac{u^4}{4}\right|_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= \frac{16-1}{4} \\ &= \frac{15}{4}. \end{aligned}$$

Note the endpoint shift using $u = x + 1$.

8. (9 points) Suppose the current in an AC circuit at time t is given by $i(t) = 2\cos(t) + 3\sin(t)$. Find the first maximum after $t = 0$. Only an exact answer will receive full credit. You do not have to specify which t value corresponds to the maximum.

Solution: Since we are finding a maximum, we differentiate: $i'(t) = -2\sin t + 3\cos t$. Setting this equal to zero yields $2\sin t = 3\cos t$, which, after a few divisions, gives $\tan t = \frac{3}{2}$. Since we are only considering the first maximum, we assume that there is a unique t value satisfying $\tan t = \frac{3}{2}$ and that this t value gives us the maximum current. We will denote it $t_{\max}$. If we think of $t_{\max}$ as an acute angle on a right triangle, $\tan t_{\max} = \frac{3}{2}$ means that the opposite side divided by the adjacent side must be $\frac{3}{2}$. Using the Pythagorean Theorem, the hypotenuse of this triangle is $\sqrt{13}$. Using the same triangle and SOHCAHTOA, we have $\sin t_{\max} = \frac{3}{\sqrt{13}}$ and $\cos t_{\max} = \frac{2}{\sqrt{13}}$. Now we can plug in to the orignal equation:

$$\begin{aligned} i_{\max} &= 2\cos t_{\max} + 3\sin t_{\max} \\ &= 2 \cdot \frac{2}{\sqrt{13}} + 3 \cdot \frac{3}{\sqrt{13}} \\ &= \frac{4}{\sqrt{13}} + \frac{9}{\sqrt{13}} \\ &= \frac{13}{\sqrt{13}} \\ &= \sqrt{13}. \end{aligned}$$

9. (8 points) Evaluate the integral: $\displaystyle\int \frac{\sqrt{\ln x}}{x}dx$

Solution: Let $u = \ln x$, so $du = \frac{1}{x}dx$. Then

$$\begin{aligned}\int \frac{\sqrt{\ln x}}{x}dx &= \int \sqrt{u}du \\ &= \int u^{1/2}du \\ &= \frac{u^{3/2}}{3/2} + C \\ &= \frac{2u^{3/2}}{3} + C \\ &= \frac{2(\ln x)^{3/2}}{3} + C.\end{aligned}$$

10. (9 points) Evaluate the integral: $\displaystyle\int \frac{e^{2r}\tan^2(e^{2r})}{\sin^2(e^{2r})}dr.$

Solution: Let $u = e^{2r}$, so $du = e^{2r} \cdot 2 \Rightarrow \frac{1}{2}du = e^{2r}dr$. Then

$$\begin{aligned}\int \frac{e^{2r}\tan^2(e^{2r})}{\sin^2(e^{2r})}dr &= \frac{1}{2}\int \frac{\tan^2(u)}{\sin^2(u)}du \\ &= \frac{1}{2}\int \frac{\frac{\sin^2 u}{\cos^2 u}}{\sin^2 u}du \\ &= \frac{1}{2}\int \frac{\sin^2 u}{\cos^2 u \cdot \sin^2 u}du \\ &= \frac{1}{2}\int \frac{1}{\cos^2 u}du \\ &= \frac{1}{2}\int \sec^2 u du \\ &= \frac{\tan u}{2} + C \\ &= \frac{\tan(e^{2r})}{2} + C.\end{aligned}$$

11. (10 points) Integrate: $\int_1^e \frac{4x-2}{x^2-x+e}dx$. Express your answer as an integer.

Solutions: Let $u = x^2 - x + e$. Then $du = 2x - 1 \Rightarrow 2du = 4x - 2$. Using this,

$$\begin{aligned}\int_1^e \frac{4x-2}{x^2-x+e}dx &= 2\int_{1^2-1+e}^{e^2-e+e} \frac{du}{u} \\ &= 2\int_e^{e^2} \frac{du}{u} \\ &= 2\ln u\Big|_e^{e^2} \\ &= 2(\ln e^2 - \ln e) \\ &= 2(2-1) \\ &= 2.\end{aligned}$$

12. (9 points) Integrate: $\int_0^{\pi/8} e^{\sin(4x)}\cos(4x)dx$. Do not round your answer.

Solution: Let $u = \sin(4x)$, so $du = \cos(4x)\cdot 4dx \Rightarrow \frac{1}{4}du = \cos(4x)dx$. Then

$$\begin{aligned}\int_0^{\pi/8} e^{\sin(4x)}\cos(4x)dx &= \frac{1}{4}\int_{\sin(4\cdot 0)}^{\sin(4\cdot\pi/8)} e^u du \\ &= \frac{1}{4}\int_0^1 e^u du \\ &= \frac{1}{4}e^u\Big|_0^1 \\ &= \frac{1}{4}(e^1 - e^0) \\ &= \frac{e-1}{4}.\end{aligned}$$

Remember that $e^0 = 1$; in fact, for any $a > 0, a \neq 1$, $a^0 = 1$.