

Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

1. Line integral of a function $f(x, y, z)$ along C , parameterized by $x = x(t)$, $y = y(t)$, $z = z(t)$ and $a \leq t \leq b$, is

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

(independent of orientation of C , other properties and applications of line integrals of f)

Remarks:

- (a) $\int_C f(x, y, z) ds$ is sometimes called the “line integral of f with respect to arc length”
- (b) $\int_C f(x, y, z) dx = \int_a^b f(x(t), y(t), z(t)) x'(t) dt$
- (c) $\int_C f(x, y, z) dy = \int_a^b f(x(t), y(t), z(t)) y'(t) dt$
- (d) $\int_C f(x, y, z) dz = \int_a^b f(x(t), y(t), z(t)) z'(t) dt$

2. Line integral of vector field $\vec{F}(x, y, z)$ along C , parameterized by $\vec{r}(t)$ and $a \leq t \leq b$, is given by

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt.$$

(depends on orientation of C , other properties and applications of line integrals of f)

3. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \vec{T}) ds$$

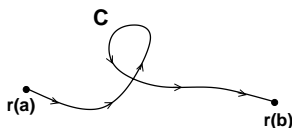
where \vec{T} is the unit tangent vector to the curve C .

4. If $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz;$$

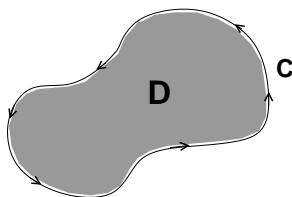
$$\text{Work} = \int_C \vec{F} \cdot d\vec{r}.$$

5. FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS: $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)):$



6. A vector field $\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$ is *conservative* (i.e. $\vec{F} = \nabla f$) if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{F}(\vec{x}) = \nabla f(\vec{x})$.

7. GREEN'S THEOREM: $\int_C P(x, y) dx + Q(x, y) dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$ ($C =$ boundary of D):



As a consequence of Green's Theorem one has

$$\frac{1}{2} \int_C x dy - y dx = \int_C x dy = - \int_C y dx = \text{Area}(D)$$

8. Del Operator: $\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$; if $\vec{F}(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$, then

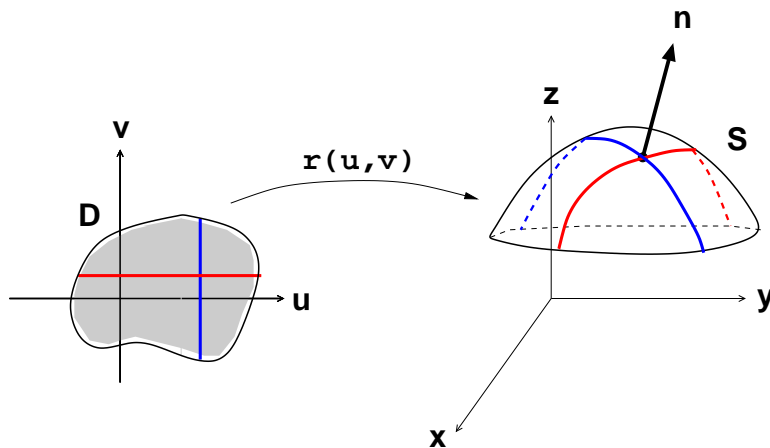
$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

(i) If $\text{curl } \vec{F} = \vec{0}$, then \vec{F} is a conservative vector field (i.e., $\vec{F}(\vec{x}) = \nabla f(\vec{x})$).

(ii) If $\text{curl } \vec{F} = \vec{0}$, then \vec{F} is *irrotational*; if $\text{div } \vec{F} = 0$, then \vec{F} is *incompressible*.

9. Parametric surface S : $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$, where $(u, v) \in D$:



Normal vector to surface S : $\vec{n} = \vec{r}_u \times \vec{r}_v$; tangent planes and normal lines to parametric surfaces.

10. Surface area of a surface S :

(i) $A(S) = \iint_D |\vec{r}_u \times \vec{r}_v| dA$

(ii) If S is the graph of $z = h(x, y)$ above D , then $A(S) = \iint_D \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} dA$;

Remark: $dS = |\vec{r}_u \times \vec{r}_v| dA =$ differential of surface area; while $d\vec{S} = (\vec{r}_u \times \vec{r}_v) dA$

11. The surface integral of $f(x, y, z)$ over the surface S :

(i) $\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) |\vec{r}_u \times \vec{r}_v| dA.$

(ii) If S is the graph of $z = h(x, y)$ above D , then

$$\iint_S f(x, y, z) dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + (\partial h/\partial x)^2 + (\partial h/\partial y)^2} dA.$$

12. The surface integral of \vec{F} over the surface S (recall, $d\vec{S} = (\vec{r}_u \times \vec{r}_v) dA$):

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n}) dS = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) dA.$$

If S is the graph of $z = h(x, y)$ above D , with \vec{n} oriented upward, and $\vec{F} = \langle P, Q, R \rangle$, then

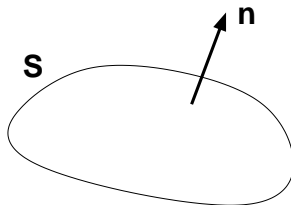
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left(-P \frac{\partial h}{\partial x} - Q \frac{\partial h}{\partial y} + R \right) dA.$$

(i) Connection between surface integral of a vector field and a function:

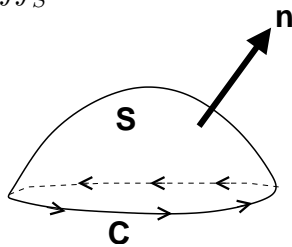
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n}) dS.$$

(The above gives another way to compute $\iint_S \vec{F} \cdot d\vec{S}$)

(ii) $\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n}) dS =$ flux of \vec{F} across the surface S .



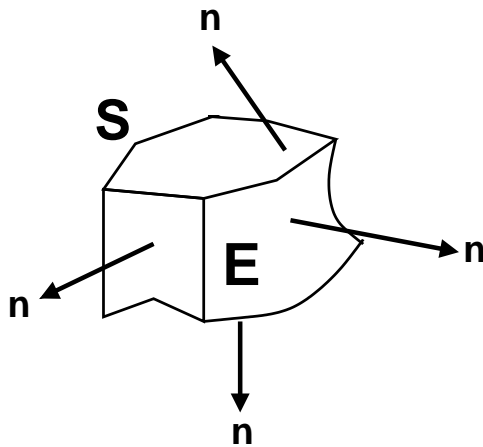
13. STOKES' THEOREM: $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$ (recall, $\text{curl } \vec{F} = \nabla \times \vec{F}$).



$\int_C \vec{F} \cdot d\vec{r} = \text{circulation of } \vec{F} \text{ around } C.$

14. THE DIVERGENCE THEOREM/GAUSS' THEOREM: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \, dV$

(recall, $\text{div } \vec{F} = \nabla \cdot \vec{F}$).



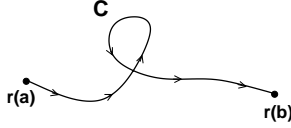
15. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \vec{\mathbf{r}}(t), \text{ where } a \leq t \leq b$	$S : \vec{\mathbf{r}}(u, v), \text{ where } (u, v) \in D$
$ds = \vec{\mathbf{r}}'(t) dt = \text{differential of arc length}$	$dS = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v dA = \text{differential of surface area}$
$\int_C ds = \text{length of } C$	$\iint_S dS = \text{surface area of } S$
$\int_C f(x, y, z) ds = \int_a^b f(\vec{\mathbf{r}}(t)) \vec{\mathbf{r}}'(t) dt$ (independent of orientation of C)	$\iint_S f(x, y, z) dS = \iint_D f(\vec{\mathbf{r}}(u, v)) \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v dA$ (independent of normal vector $\vec{\mathbf{n}}$)
$d\vec{\mathbf{r}} = \vec{\mathbf{r}}'(t) dt$	$d\vec{\mathbf{S}} = (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) dt$ (depends on orientation of C)	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_D \vec{\mathbf{F}}(\vec{\mathbf{r}}(u, v)) \cdot (\vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v) dA$ (depends on normal vector $\vec{\mathbf{n}}$)
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}}) ds$ The <i>circulation</i> of $\vec{\mathbf{F}}$ around C	$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S (\vec{\mathbf{F}} \cdot \vec{\mathbf{n}}) dS$ The <i>flux</i> of $\vec{\mathbf{F}}$ across S in direction $\vec{\mathbf{n}}$

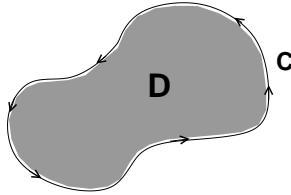
16. Integration Theorems:

FUNDAMENTAL THEOREM OF CALCULUS: $\int_a^b F'(x) dx = F(b) - F(a)$

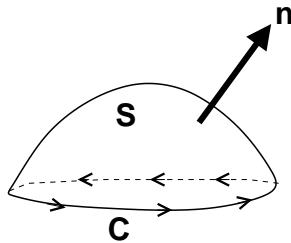
FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS: $\int_a^b \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$



GREEN'S THEOREM: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P(x, y) dx + Q(x, y) dy$



STOKES' THEOREM: $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$



DIVERGENCE THEOREM: $\iiint_E \text{div } \vec{F} dV = \iint_S \vec{F} \cdot d\vec{S}$

