Summer MA 15200 Lesson 19 Section 2.6, Section 2.7 (part 1)

I The Domain of a Function

Remember that the domain is the set of x's in a function, or the set of 'first things'. For many functions, such as $f(x)=2x-3$, x could be replaced with any real number to find $f(x)$. Therefore the domain could be written in set-builder notation as $\{x \mid x \text{ is a real number}\}$ or $(-\infty,\infty)$ using interval notation. However, not all functions have domains that are all real numbers. In real life examples, the domain is often **limited** to numbers that only make sense. For example, if $d=50t$ (distance in miles equals 50 times time in hours), the time can only be 0 or positive numbers. No one would travel negative hours. So the domain would be $\{t \mid t \geq 0\}$ or $[0,\infty)$.

Examine the following two functions.

$f(x)=\sqrt{4-x}$ $\qquad\qquad g(x)=\dfrac{2}{x-3}$

If x is replaced with a 5 in function f, the result would be $\sqrt{-1}$. While we've discussed imaginary numbers, when speaking of domains, we only think of real numbers. So we know 5 is not in the domain of f. How do we find the domain? We know that we can only take the square root of a positive number or zero. Therefore 4 – x must be greater than or equal to zero.

$$4-x \geq 0$$
$$-x \geq -4 \qquad \text{The domain of } f \text{ is } \{x \mid x \leq 4\} \text{ or } (-\infty,4].$$
$$x \leq 4$$

In function g, if x is replaced with 3, the result would be $\frac{2}{0}$, which is an undefined number. 3 is not in the domain of function g. We can write the domain as $\{x \mid x \text{ is a real number and } x \neq 3\}$ or $(-\infty,3)\cup(3,\infty)$.

The above examples illustrate the 'problems' you will encounter possibly when determining domain of functions. Otherwise the domain of functions will probably be all real numbers.

1. **If the function has a square root, set the radicand greater than or equal to zero and solve. The result is the domain.**
2. **If the function has a denominator, set the denominator to zero and solve. The number(s) that result(s) are excluded from the domain.**

Ex 1: Find the domain of each function. Write each using set-builder notation and interval notation.

$a)\quad f(x) = 3x^2 + 2x - 3$

$b)\quad g(x) = \sqrt{2x+3}$

$c)\quad h(x) = \dfrac{3x}{x^2 - 5x + 6}$

II The Algebra of Functions

Functions can be combined using addition, subtraction, multiplication, or division on the right sides of two given functions. The domain of the function that results will be all real numbers common to both original domains and, in case of $\dfrac{f(x)}{g(x)}$, numbers that do not make a zero denominator.

Here are the definitions of these functions given two functions $f(x)$ and $g(x)$.

1. Sum Function: $(f+g)(x) = f(x) + g(x)$
2. Difference Function: $(f-g)(x) = f(x) - g(x)$
3. Product Function: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient Function: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

Basically, you are adding, subtracting, multiplying, or dividing the function values or y values.

Ex 2: Given: $f(x) = 2x^2 - 4x + 3$ and $g(x) = 3x - 4$, find the following functions. State the domain of each.

$a)\quad (f+g)(x) =$

$b)\quad (g-f)(x) =$

$c)\quad \left(\dfrac{f}{g}\right)(x) =$

$d)\quad (gf)(x) =$

Ex 3: Given: $f(x) = 3x + 2$ and $g(x) = \sqrt{x+2}$, find the following function values.

$a)$ $(f+g)(2) =$

$b)$ $(f \cdot g)(8) =$

$c)$ $\left(\frac{g}{f}\right)(10) =$

$d)$ $\left(\frac{f}{g}\right)(-2) =$

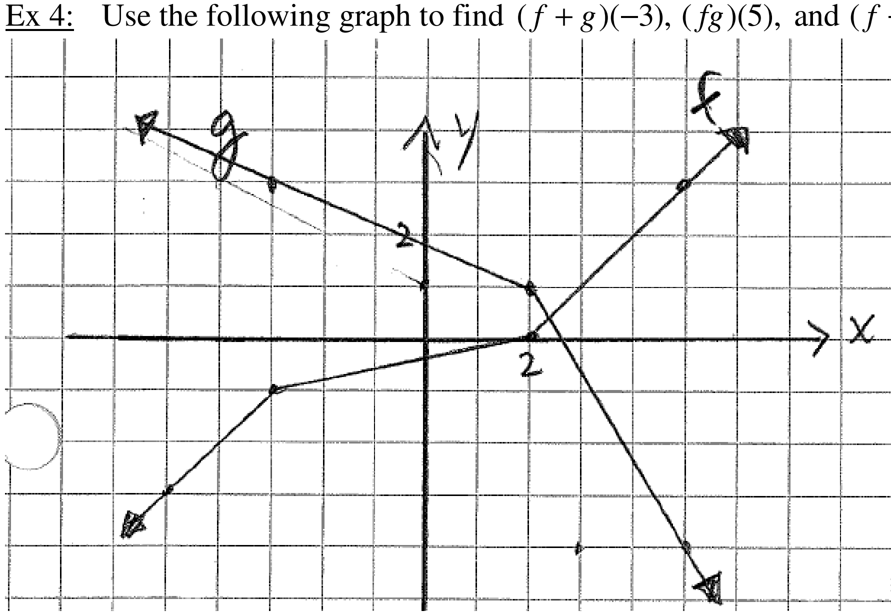
Note: In the case of the quotient function, determine the domain **before** trying to simplify the quotient.

$f(x) = x^2 - 4 \qquad g(x) = x + 2$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x^2-4}{x+2} = \frac{(x+2)(x-2)}{x+2} = x - 2$$

While the domain of expression $x - 2$ would be all reals, you must consider there was a denominator and $x \neq -2$. The domain of the quotient function is $(-\infty, -2) \cup (-2, \infty)$.

Ex 4: Use the following graph to find $(f+g)(-3)$, $(fg)(5)$, and $(f-g)(2)$.



III Composite Functions

There is another way to combine functions called a composition of the functions. A good way to illustrate this type of function is the following. Suppose that a store is having a sale. There is a computer that regularly cost $600 on sale for $100 off. However, they are also offering a discount of 10% off (pay 90%), even on sale items. So, first $100 is subtracted from the computer, then 90% of that price if determined. If x could be thought of as the price of the computer... original price: $P(x) = x - 100$, then sale price: $S(P(x))=0.90(P(x)) = 0.90(x - 100)$. Essentially, one function value is put into a second function.

The Composition of Functions:
The **composition of the function *f* with *g*** is denoted by $f \circ g$ and is defined by

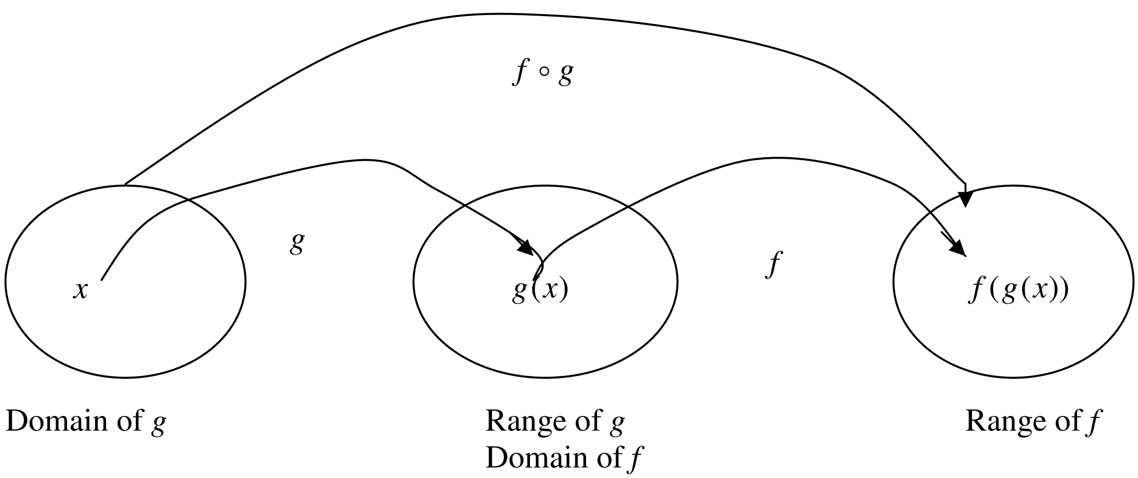
$$(f \circ g)(x) = f(g(x)).$$

The **domain of the composite function $f \circ g$** is the set of all x such that

1. x is in the domain of g and
2. $g(x)$ is in the domain of f.

For $(f \circ g)(x) = f(g(x))$ $g(x)$ is called the 'inner function' and $f(x)$ is the 'outer function'.

The following picture illustrates the composition of two functions.



Ex 4: Given: $f(x) = 2x - 3$, $g(x) = x^2 - 2x$, and $h(x) = \sqrt{x}$, find the following values.

$a)$ $(f \circ g)(-2) =$

$b)$ $(g \circ f)(-2) =$

$c)$ $(g \circ h)(4) =$

$d)$ $(h \circ g)(1) =$

To find the domain of a composite function $f \circ g$, ask yourself these questions?

- What is the domain of $g(x)$ (inner function)?
- For which of these numbers would $g(x)$ be in the domain of f?

Ex 5: Find the composite functions $(f \circ g)(x)$ *and* $(g \circ f)(x)$ and the domains of the composite functions given....

$a)$ $f(x) = \sqrt{x+1}$, $g(x) = x + 3$

$b)$ $f(x) = 2x - 4$, $g(x) = \sqrt{x}$

$c)\quad f(x)=\sqrt{x},\ g(x)=x^2+2$

IV Decomposing Functions

Suppose $h(x)=(2+x)^3$. This function takes 2 + *x* and raised it to the cube power. We could write $h(x)=f(g(x))=(f\circ g)(x)$ where $f(x)=x^3$ and $g(x)=2+x$.

Ex 6: Find two functions *f* and *g*, such that $h(x)=(f\circ g)(x)$ for the functions below. Note: *f* is the outer function and *g* is the inner function.

$a)\quad h(x)=8x^2-19$

$b)\quad h(x)=(2x+3)^3$

$c)\quad h(x)=\dfrac{2}{x-4}$

$d)\quad h(x)=\sqrt[3]{x+5}$

V Inverse functions

These equations represent **inverse functions**. Examine the temperature formulas below.

$$C = \frac{5}{9}(F - 32) \qquad\qquad F = \frac{9}{5}C + 32$$

Replace F in the first equation with 50º.

$$C = \frac{5}{9}(50 - 32)$$
$$C = \frac{5}{9}(18)$$
$$C = 10$$

Now, replace C in the second with 10º.

$$F = \frac{9}{5}(10) + 32$$
$$F = 18 + 32$$
$$F = 50$$

Notice these functions do opposite things. The first equation turned 50°F to 10°C. The second equation turned 10°C back to 50°F. Such functions are called <u>inverse functions.</u>

Here are two other functions that are inverses of each other.

$$y = 2x + 3 \, and \; y = \frac{x-3}{2}$$

Notice that in the first function a number is multiplied by 2, then 3 is added to the result. In the second function 3 is subtracted from a number, then divided by 2. Begin with 5 as the number in the first function. Multiply by 2, and then add 3; the result is 13. Now, let 13 be the number in the second function. Subtract 3, divide by 2; the result is 5 (the original number selected for the first function).

Let the first equation above be $f(x) = 2x + 3$ and the second be $g(x) = \frac{x-3}{2}$. Examine:

$$f(g(x)) = f\left(\frac{x-3}{2}\right) = 2\left(\frac{x-3}{2}\right) + 3 = x - 3 + 3 = x$$

$$g(f(x)) = g(2x+3) = \frac{2x+3-3}{2} = \frac{2x}{2} = x$$

The composition of two inverse functions of *x* will always be *x*!

VI Verifying Inverse Functions

As demonstrated above, when two functions are inverses of each other, their composition functions (both $f(g(x))$ and $g(f(x))$ equal *x*. To verify (or prove) that two functions are inverses, find both compositions and show they equal *x*.

Ex 7: Determine if the following functions are inverses of each other.

$a)\quad f(x)=4x+5,\ g(x)=\frac{x-5}{4}$

$b)\quad f(x)=\frac{x+1}{x},\ g(x)=\frac{1}{x+1}$

$c)\quad f(x)=\frac{1}{x+1},\quad g(x)=x+1$