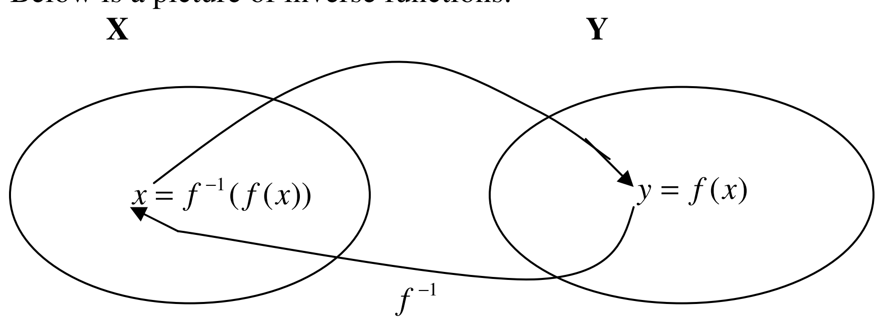
Summer MA 15200 Lesson 20 Section 2.7 (part 2), Section 4.1

Definition of the Inverse of a Function:
Let *f* and *g* be two functions such that $f(g(x)) = x$ for every *x* in the domain of *g* and $g(f(x)) = x$ for every *x* in the domain of *f*.
The function *g* is called the **inverse of function *f*** and is denoted by f^{-1} (read *f* inverse).
Therefore $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. The domain of *f* is equal to the range of f^{-1}, and vice-versa.

Note: The notation f^{-1} does **not** mean $\frac{1}{f}$. The -1 is not an exponent, it is a notation.

Below is a picture of inverse functions.



There is also a good picture in Figure 2.54 on page 285 of the textbook.

I Finding the Inverse of a Function

To find the inverse of a function *f* follow these steps.

1. Replace $f(x)$ with *y* in the equation.
2. Interchange *x* and *y*.
3. Solve for *y*. If this equation does not define *y* as a function of *x*, the function *f* does not have an inverse function and this procedure ends. If this equation does define *y* as a function of *x*, the function *f* has an inverse function.
4. If the function does have an inverse, replace the *y* in step 3 with $f^{-1}(x)$.

<u>Ex 1:</u> Find the inverse of each function or verify that it does not have an inverse.

$a)$ $f(x) = 2x - 3$

$b)$ $f(x) = \dfrac{2}{1+x}$

$c) \quad f(x) = \dfrac{x+2}{x}$

$d) \quad f(x) = \dfrac{1}{x} + 5$

$e) \quad f(x) = x^3 + 2$

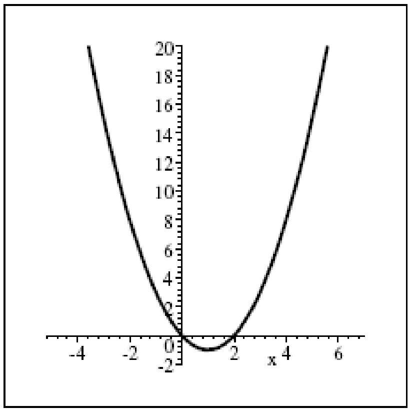
II 1-1 Functions

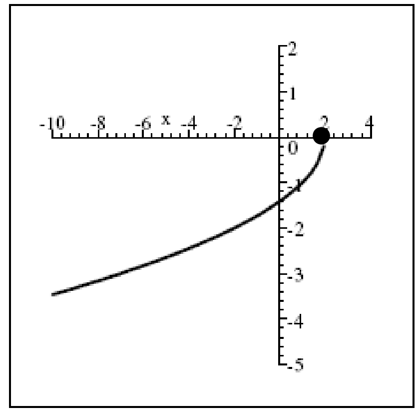
In order for a function to have an inverse, it must be a **1-to-1 function**, which means for each *x* in the domain, there is only 1 *y* in the range (definition of function) ***and*** for each *y* in the range, there is only 1 *x* in the domain.

One-to-one Functions: A function *f* from a set X to a set Y is called one-to-one if and only if different numbers in the domain of *f* have different outputs in the range of *f*. If the graph of the function *f* is known, the graph must pass not only the vertical line test, but the **horizontal line test** as well.

<u>A function is only one-to-one (1-1) if it is constantly increasing or constantly decreasing.</u>

Ex 2: Determine if these graphs represent 1-to-1 functions.

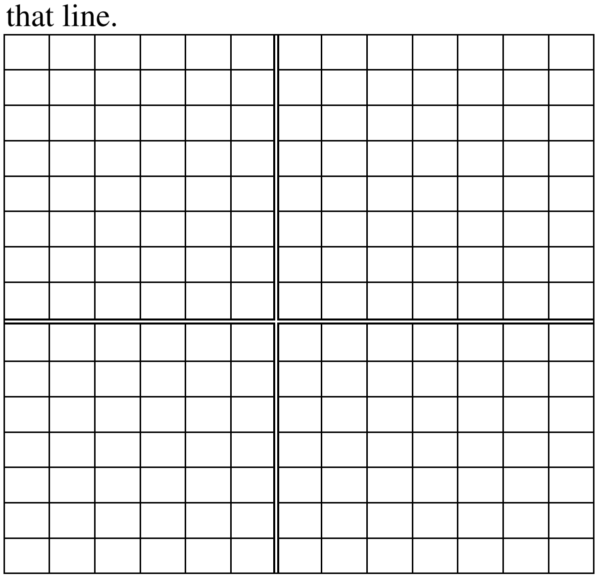
a) 

b) 

III Graphs of Inverse Functions

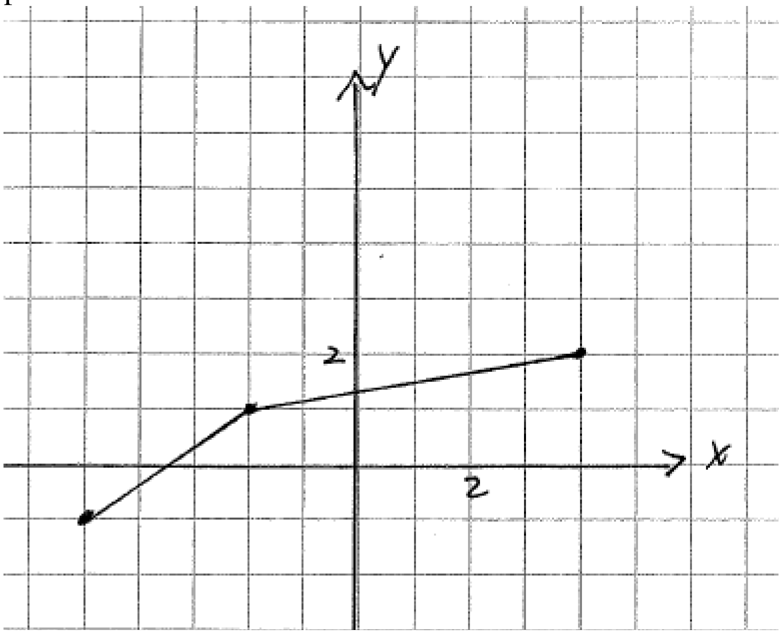
There is a relationship between the graph of a 1-1 function *f* and its inverse f^{-1}. If the ordered pair (a,b) is on the graph of function *f*, the ordered pair (b,a) would be on the graph of function f^{-1}. These points would be symmetric with respect to the line $y=x$. The graph of f^{-1} is a **reflection of the graph of *f* about the line *y* = *x***. (symmetry about the line $y=x$)

Ex 3: Graph the function $f(x)=-2x+4$ Use key points of that graph to sketch the graph of the inverse of the function. Sketch in the line $y = x$ to show the symmetry about that line.



In inverse functions:
(1) *x* and *y* values of each ordered pair are reversed.
(2) Domain and range are reversed.

Ex 4: Use the graph of the function below to sketch its inverse on the same coordinate plane.



Section 4.1

We have discussed powers where the exponents are integers or rational numbers. There also exists powers such as $2^{\sqrt{3}}$. You can approximate powers on your calculator using the power key. On most one-liner scientific calculators, the power key looks like $\boxed{y^x}$

Enter the base into the calculator first, press the power key, enter the exponent, and press enter or equal.

Ex 5: Approximate the following powers to 4 decimal places.

$a)\quad 2^{3\sqrt{2}} =$

$b)\quad (2.3)^{4.8} =$

IV Exponential Functions

An **exponential function *f* with base *b*** is defined by $f(x) = b^x$ or $y = b^x$, where b is a positive constant other than 1 and x is any real number.

A calculator may be needed to evaluate some function values of exponential functions. (See example 5 above.)

The following are **not** exponential functions. Why?

$f(x) = x^3$ $\quad f(x) = 1^x$

$f(x) = (-4)^x$ $\quad f(x) = x^x$

Many real life situations model exponential functions. One example given in your textbook models the average amount spent(to the nearest dollar) at a shopping mall after x hours and is $f(x) = 42.2(1.56)^x$. The base of this function is 1.56. Notice there is also a 'constant' (42.2) multiplied by the power. Be sure to follow the order of operations; find the exponent power first, then multiply that answer by the 42.2.

Suppose you wanted to find the amount spent in a mall after browsing for 3 hours.
Let $x = 3$.

$$f(3) = 42.2(1.56)^3$$
$$= 42.2(3.796416)$$
$$= 160.2087552$$

To the nearest dollar, a person on average would spend \$160.

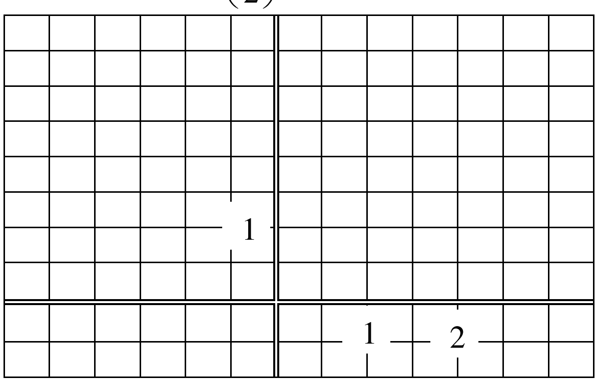
V Graphing Exponential Functions

To graph an exponential function, make a table of ordered pairs as you have for other types of graphs. Notice: If $x = 0$ for b^x, the value is 1 (zero power is 1). For a basic exponential function, the y-intercept is 1.

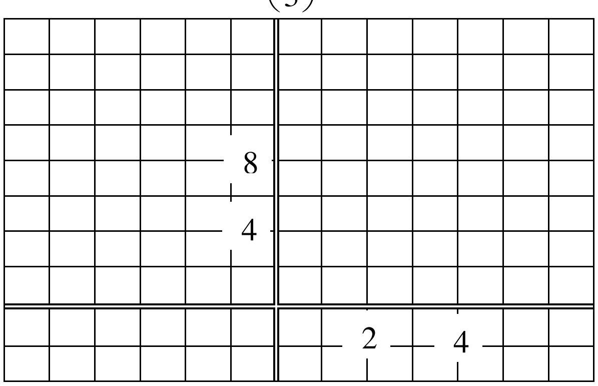
Also, notice that y values will always be positive, so the graph always lies above the x-axis.

Ex 6: Graph each exponential function.

$a)\quad y = \left(\frac{3}{2}\right)^x$



$b)\quad f(x) = \left(\frac{1}{3}\right)^x$



There are several exponential graphs shown in figure 4.4 on page 415 of the text. After examining several graphs, the following characteristics can be found.

Characteristics of Exponential Functions of the form $f(x) = b^x$

1. The domain of the function is all real numbers $(-\infty, \infty)$ and the range is all positive real numbers $(0, \infty)$ (graph always lies above the x-axis).
2. Such a graph will always pass through the point (0, 1) and the y-intercept is 1. There will be no x-intercept.
3. If the base b is greater than 1 $(b > 1)$, the graph goes up to the right and is an increasing function. The greater the value of b, the steeper the increase (exponential growth).
4. If the base is between 0 and 1 $(0 < x < 1)$, the graph goes down to the right and is a decreasing function (exponential decay). The smaller the value of b, the steeper the decrease.
5. The graph represents a 1-1 function and therefore will have an inverse.
6. The graph approaches but does not touch the x-axis. The x-axis is known as an asymptote.

VI The Natural Base *e* and the Natural Exponential Function

There is an irrational number, whose symbol is e, that is used quite often for exponential function's base. This number is the value of $\left(1+\frac{1}{n}\right)^n$ as n becomes very, very large or goes to infinity. An approximation of this number is $e = 2.718281827$ and the number e is called the **natural base**. The function $f(x) = e^x$ is called the **natural exponential function.** To approximate the powers of e, use these steps on your calculator.

1. Enter the exponent in your calculator.
2. Because the e power is above the *ln* key, you must press the $\boxed{\text{2nd}}$ key first and then the $\boxed{\text{ln}}$ key.
3. The value is approximately the power.

<u>Ex 7:</u> Approximate each power to 4 decimal places.

$a)\quad e^3 =$

$b)\quad e^{0.024} =$

$c)\quad e^{-\frac{2}{3}} =$

Another life model that uses an exponential function is $f(x) = 1.26e^{0.247x}$, which approximates the gray wolf population of the Northern Rocky Mountains x years after 1978. (Notice: Multiply 0.247 by x, find the number e to that power, then multiply the result by 1.26.)

<u>Ex 8</u>: Use the model above the approximate the gray wolf population in 2008.
2008 is 30 years after 1978. Let $x = 30$.

$$\begin{aligned} f(30) &= 1.26e^{0.247(30)} \\ &= 1.26e^{7.41} \\ &= 1.26(1652.426347) \\ &\approx 2082 \end{aligned}$$

2082 gray wolves

VII Compound Interest

One of the most common models of exponential functions used in life are the models of compound interest. You know that the Simple Interest Formula is $I = P\,r\,t$ and the amount accumulated with simple interest is $A = P + P\,r\,t$. However, in this model, interest is only figured at the very end of the time period. In most situations, interest is determined more often; sometimes annually, monthly quarterly, etc. Then the amount accumulated becomes the formula on the next page.

Compound Interest Formula:
If an account has interest compounded n times per year for t years with principal P and an annual interest rate r (in decimal form), the amount of money in the account is found by
$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Some banks or financial institutions may **compound interest continuously**. If that happens, the formula above becomes the following that uses the number e.

Compound Continuously Formula:
If an account is compounded continuously for t years with principal P at an annual intrest rate r (in decimal form), the amount of money in the account is found by
$A = Pe^{rt}$.

In lesson 28 we will discuss these compound interest formulas more.

Ex 9: Suppose \$8000 is invested for 5 years at 4.5% annual interest. Find the amount in the account at the end of the 5 years if...

a) interest is compounded quarterly

Always convert percent rates to decimals in these types of formulas. We are also assuming no additional deposits were made.

b) interest is compounded monthly

c) interest is compounded continuously

Ex 10: Lily's parents deposited an amount in her account on her day of birth. The account earned 6% annual interest compounded continuously and on her 18th birthday the account was \$40,000. How much was the initial deposit by her parents?

Ex 11: Which investment would yield the greatest amount of money for an initial investment of \$500 over a period of 6 years; 7% compounded quarterly or 6% compounded continuously?

VIII Other Applied Problems

Ex 12: The population of a city is 45,000 in 2000. The population growth is represented by $P = 45e^{0.011t}$ in thousands for t years after 2000. What will be the population in 2010?

Ex 13: The formula $S = C(1+r)^t$ models an inflation value for t years from now, where C is the current price, r is the inflation rate, and S is the inflated value. If a house currently is worth \$89,000 and the inflation rate is 1.2%, what would the house be worth in 15 years from now?

Ex 14: Sometimes more than one function model could be used for some life situations. Suppose the Purdue Mathematics department determines that the percentage of mathematics remembered x weeks after learning the math can be described by the linear model below or the exponential model below.

$$f(x) = -3.6x + 87$$

$$g(x) = 78e^{-0.1x} + 22$$

a) Determine the percentage of math remembered 4 weeks after learning the math using the linear model.

b) Determine the percentage of math remembered 4 weeks after learning the math using the exponential model.

c) If statistics show that, on average, a math student remembered 75% of what they learned 4 weeks after learning, which model best approximated the percentage after 4 weeks?