Rational Functions

A rational function is a function where the numerator and denominator are polynomials. Examples of rational functions are:

$$f(x)=\frac{2x-3}{x^2+3x+1} \qquad g(x)=\frac{4x-3}{2x} \qquad h(x)=\frac{3x^4-2x^2}{3x+9}$$

Vertical asymptotes: The line <u>$x = a$</u> is a **vertical asymptote** if <u>$f(x) \longrightarrow \infty$ or $f(x) \longrightarrow -\infty$</u> as x approaches a from either the left or the right.

Vertical asymptotes occur at values of *x* that cause zero denominators.

Horizontal asymptotes: The line <u>$y = c$</u> is a **horizontal asymptote** for the graph of the function f if <u>$f(x) \longrightarrow c$ as $x \longrightarrow \infty$ or as $x \longrightarrow -\infty$</u>

Horizontal asymptotes occur whenever the ratio of the numerator to the denominator equals a constant number as *x* gets very large or very small.

Vertical asymptotes are never 'crossed'. However, horizontal asymptotes may be crossed in the middle. The graph only approaches the horizontal line as *x* goes to infinity or negative infinity.

Theorem on Horizontal Asymptotes: Given: $\dfrac{ax^n+bx^{n-1}+\cdots+cx+d}{ex^k+fx^{k-1}+\cdots+gx+h}$

Note: Usually numerators or denominators will be monomials, binomials, or trinomials. The notation above is simply a general notation to say a polynomial over another polynomial. Let n be the highest exponent in the numerator polynomial and k be the highest exponent in the denominator. Then the following statements are true.

1) If <u>$n < k$</u>, then <u>*the x-axis (y = 0)*</u> is the horizontal asymptote.
2) If <u>$n = k$</u>, then $y = \dfrac{a}{e}$ is the horizontal asymptote.
3) If <u>$n > k$</u>, then <u>*the graph has no*</u> horizontal asymptote.

Case 1) If $n < k$, the numerator will be so much smaller than the denominator. A fraction that has a super large denominator compared to the numerator is very close to zero. For example: $\dfrac{2x-3}{4x^2+8}$ Let x become really large, such as a billion

The ratio $\dfrac{2x}{4x^2}=\dfrac{2{,}000{,}000{,}000}{4(1{,}000{,}000{,}000)^2}\approx 0$

Case 2) If $n = k$, the ratio $\frac{ax^n}{ex^k}$ will approach the ratio $\frac{a}{e}$.

For example: $\frac{3x^2}{4x^2}$ Let x approach a billion.

$$\frac{3(1{,}000{,}000{,}000)^2}{4(1{,}000{,}000{,}000)^2} = \frac{3}{4}$$

Case 3) If $n > k$, the numerator is growing much faster than the denominator and the ratio will go to a super big number.

For example: $\frac{3x^3 + 2x}{3x - 1}$ Let x approach a billion

$$\frac{3(1{,}000{,}000{,}000)^3}{3(1{,}000{,}000{,}000)} = billion^2 \approx \infty$$

1) Find the vertical asymptote(s) for the graph of f,

$$f(x) = \frac{2x+1}{x^2+5x+6} \qquad f(x) = \frac{6x^2+2x}{2x^2-9x-5}$$

2) Find the horizontal asymptote for the graph of f, if it exist.

$$f(x) = \frac{6x-1}{3x^2+4x-3} \qquad f(x) = \frac{6x^2+5x+1}{3x+2} \qquad f(x) = \frac{6x^2+7x+1}{3x^2-5x+2}$$

Rational Functions

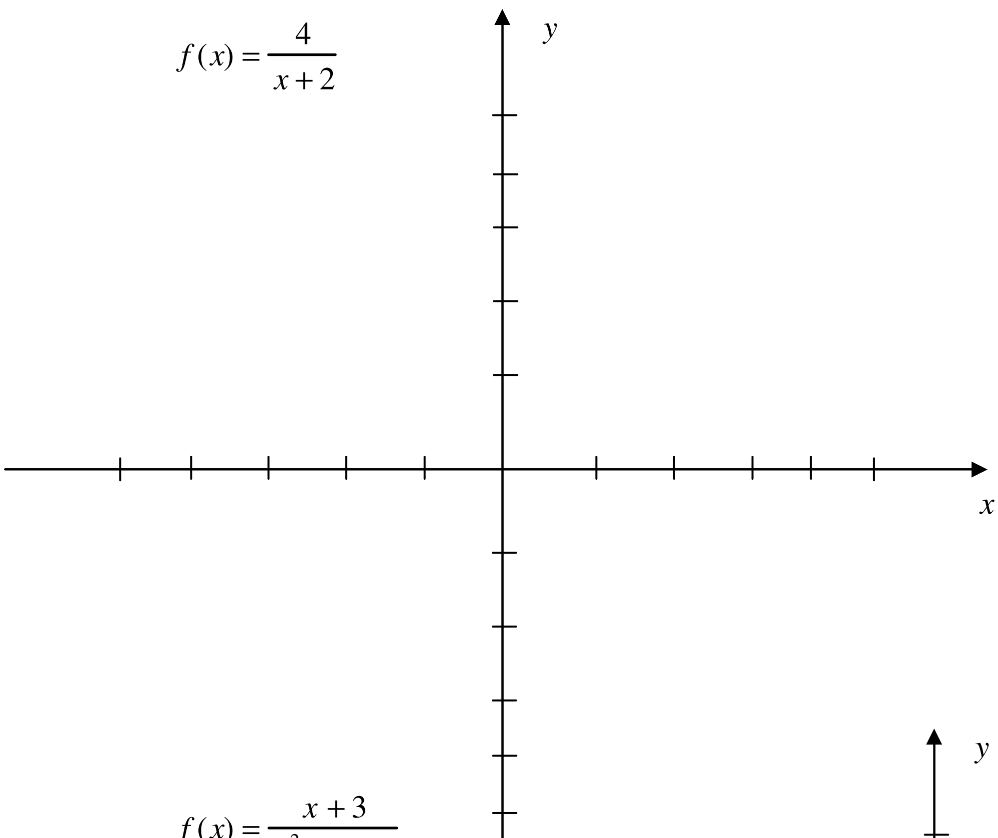
For each function f:

a) Find the horizontal asymptote, if it exists.
b) Find the vertical asymptote(s), if any exist.
c) x-intercept(s), if any exist. (What values of x make the numerator be zero?)
d) y-intercept, if it exists. (Let $x = 0$)
e) Sketch the graph of f.

> This information is easy to find and will 'frame' the graph of the function.

Also: Find the domain and range and the intervals of f when the function is increasing and decreasing.

$$f(x) = \frac{4}{x+2}$$



Sign Chart:

$$f(x) = \frac{x+3}{x^2+x-2}$$

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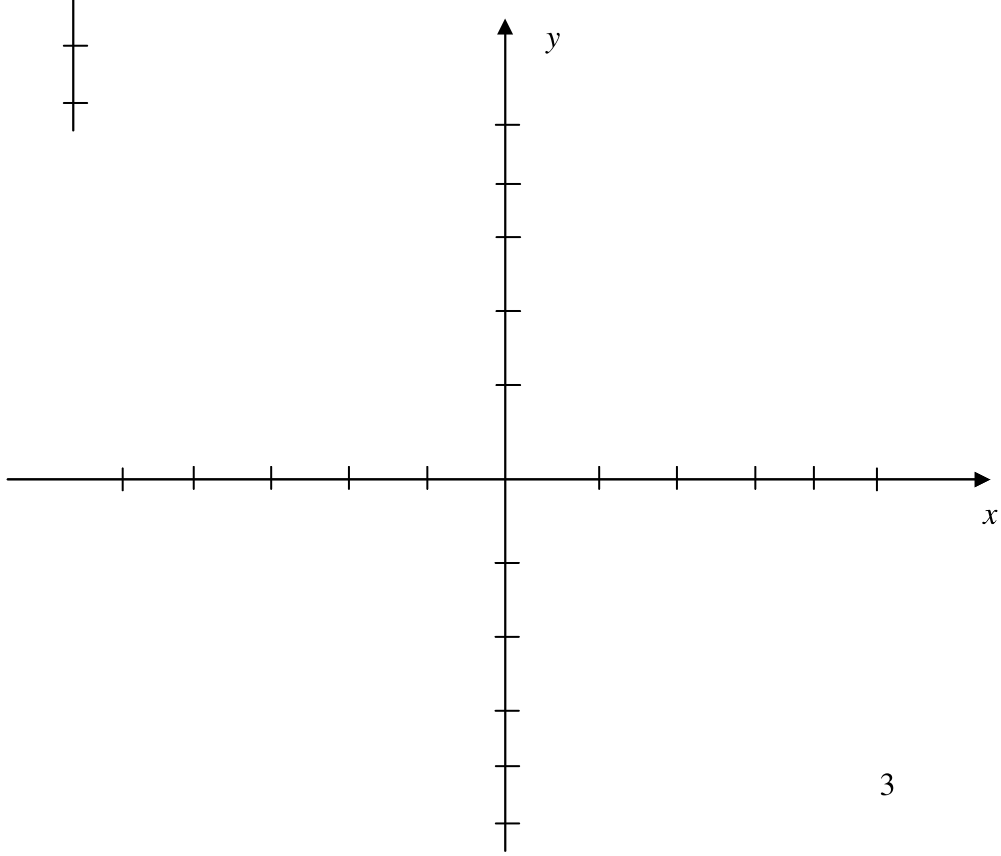
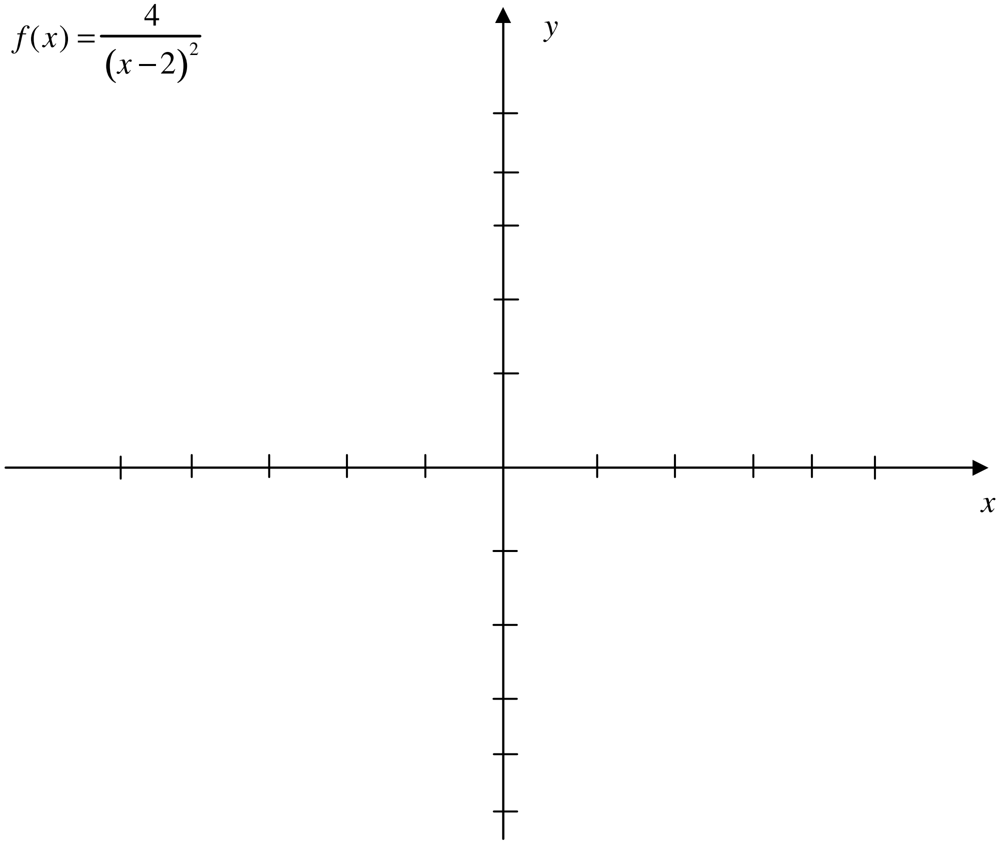


$$f(x) = \frac{4}{(x-2)^2}$$



$$f(x) = \frac{6x}{2x+1}$$