

Notes

Quantitative Analysis Questions

- Can I imagine the situation, as though I am acting it out?
- What quantities are involved? What do I know about their nature? Do some change over time? Do some stay constant?
- What is the *quantity of interest*? That is, what quantity, and its value, am I asked to find or describe?
- How could making a diagram (or a sequence of diagrams) help reveal relationships involving the quantity of interest and other quantities?
- What quantities and values do I know that could help me find the value of the quantity of interest?
- What quantities and values can I derive to help me find the value of the quantity of interest?

And so on.

The point is that you need to be specific when doing a quantitative analysis of a given situation. How specific? It is difficult to say in general, because it will depend on the situation. Use your common sense in analyzing the situation. When you first read a problem, avoid trying to think about numbers and the operations of addition, subtraction, multiplication, and division. Instead, start out by posing the questions such as those listed above and try to answer them. Once you've done that, you will have a better understanding of the problem, and thus you will be well on your way to solving it. Keep in mind that understanding the problem is the most difficult aspect of solving it. Once you understand the problem, *what to do to solve it* often follows quite easily.

**ACTIVITY 3** Down the Drain

Here is another problem situation on which to practice quantitative analysis.

► Water is flowing from a faucet into an empty tub at 4.5 gallons per minute. After 4 minutes, a drain in the tub is opened, and the water begins to flow out at 6.3 gallons per minute. ◀

- Will the tub ever fill up completely without overflowing?
- Will it ever empty completely?
- What if the faucet is turned off after 4 minutes?
- What if the rates of flow in and out are reversed?
- What assumptions do we have to make in order to answer these questions? ●

The following problem is a common, though not always popular, problem type to solve in algebra classes. Although you may be able to solve the following problem using an algebraic solution, use quantitative reasoning so as not to mask any real understanding of this type of problem.

**ACTIVITY 4** All Aboard!

Continental Coast Railroad provides service between Chicago, Illinois, and Birmingham, Alabama. A new passenger train leaves Chicago headed toward Birmingham at the same time that a freight train leaves Birmingham for Chicago. The trains are 583 miles apart on parallel tracks. The new passenger train travels at a constant speed of 120 miles per hour, and the freight train travels at a constant speed of 75 miles per hour.

- Where do they meet?
- Suppose the passenger train left one hour later than the freight train. Where do they meet? ●

When you carry out a quantitative analysis of a situation, the questions you ask yourself should be guided by your common sense. A sense-making approach to understanding a situation and then solving a problem is much more productive than trying to decide right away which computations, formulas, or mathematical techniques you need in order to solve

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the problem at hand. Using common sense may lead you to make some sort of a diagram. Never be embarrassed to use a diagram. Such diagrams often enhance your understanding of the situation, because they help you to think more explicitly about the quantities that are involved. Deciding what operations you need to perform often follows naturally from a good understanding of the situation, the quantities in it, and how those quantities are related to one another.

TAKE-AWAY MESSAGE . . .

You should be able to determine the quantities and their relationships within a given problem situation, and you should be able to use this information, together with drawings as needed, to solve problems. These steps are often useful: (1) List the quantities that are essential to the problem. (2) List known values for these quantities. (3) Determine the relationships involved, which is frequently done more easily with a drawing. (4) Use the knowledge of these relationships to solve the problem. This approach may not seem easy at first, but it becomes a powerful tool for understanding problem situations. This type of quantitative analysis can also be used with algebra story problems, and thus, when used with elementary school students, prepares them for algebra.

Learning Exercises for Section 1.2

- Some problems are simple enough that the quantitative structure is obvious, particularly after a drawing is made. The following problems are from a fifth-grade textbook.² For each problem given, make a drawing and then provide the answer to the problem.
 - The highest elevation in North America is Mount McKinley, Alaska, which is 20,320 feet above sea level. The lowest elevation in North America is Death Valley, California, which is 282 feet below sea level. What is the change in elevation from the top of Mount McKinley to Death Valley?
 - The most valuable violin in the world is the Kreutzer, created in Italy in 1727. It was sold at auction for \$1,516,000 in England in 1998. How old was the violin when it was sold?
 - Two sculptures are similar. The height of one sculpture is four times the height of the other sculpture. The smaller sculpture is 2.5 feet tall. How tall is the larger sculpture?
 - Aiko had \$20 to buy candles. She returned 2 candles for which she had paid \$4.75 each. Then she bought 3 candles for \$3.50 each and 1 candle for \$5.00. How much money did Aiko have then?
 - In Ted's class, students were asked to name their favorite sport. Football was the response of $\frac{1}{8}$ of them. If 3 students said football, how many students are in Ted's class?
 - The first year of a dog's life equals 15 "human years." The second year equals 10 human years. Every year thereafter equals 3 human years. Use this formula to find a 6-year-old dog's age in human years.
- These problems are from a sixth-grade textbook³ from a different series. This time, undertake a full quantitative analysis to solve each of the problems.
 - At Loud Sounds Music Warehouse, CDs are regularly priced at \$9.95 and tapes are regularly priced at \$6.95. Every day this month the store is offering a 10% discount on all CDs and tapes. Joshua and Jeremy go to Loud Sounds to buy a tape and a CD. They do not have much money, so they have pooled their funds. When they get to the store, they find that there is another discount plan just for that day—if they buy 3 or more items, they can save 20% (instead of 10%) on each item. If they buy a CD and a tape, how much money will they spend after the store adds a 6% sales tax on the discounted prices?
 - Kelly wants to fence in a rectangular space in her yard, 9 meters by 7.5 meters. The salesperson at the supply store recommends that she put up posts every 1.5 meters. The posts cost \$2.19 each. Kelly will also need to buy wire mesh to string between

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the posts. The wire mesh is sold by the meter from large rolls and costs \$5.98 a meter. A gate to fit in one of the spaces between the posts costs \$25.89. Seven staples are needed to attach the wire mesh to each post. Staples come in boxes of 50, and each box costs \$3.99. How much will the materials cost before sales tax?

- A 3. Amtrail trains provide efficient, nonstop transportation between Los Angeles and San Diego. Train A leaves Los Angeles headed toward San Diego at the same time that Train B leaves San Diego headed for Los Angeles, traveling on parallel tracks. Train A travels at a constant speed of 84 miles per hour. Train B travels at a constant speed of 92 miles per hour. The two stations are 132 miles apart. How long after they leave their respective stations do the trains meet?
4. My brother and I walk the same route to school every day. My brother takes 40 minutes to get to school and I take 30 minutes. Today, my brother left 8 minutes before I did.⁴
- How long will it take me to catch up with him?
 - Part of someone's work on this problem included $\frac{1}{30} - \frac{1}{40}$. What quantities do the two fractions in $\frac{1}{30} - \frac{1}{40}$ represent?
 - Suppose my brother's head start is 5 minutes instead of 8 minutes. Now how long would it take me to catch up with him?
5. At one point in a Girl Scout cookie sales drive, region C had sold 1500 boxes of cookies, and region D had sold 1200 boxes of cookies. If region D tries harder, they can sell 50 more boxes of cookies every day than region C.
- How many days will it take for region D to catch up?
 - If sales are stopped after 8 more days, can you tell how many total boxes each region sold? Explain.
- A 6. The last part of one triathlon is a 10K (10 kilometers or 10,000 meters) run. When runner Aña starts this last running part, she is 600 meters behind runner Bea. But Aña can run faster than Bea: Aña can run (on average) 225 meters each minute, and Bea can run (on average) 200 meters each minute. Who wins, Aña or Bea? If Aña wins, when does she catch up with Bea? If Bea wins, how far behind is Aña when Bea finishes?
7. Research on how students solve word problems contained the following incident.⁵ Dana, a seventh-grader in a gifted program in mathematics, was asked to solve the following problem:

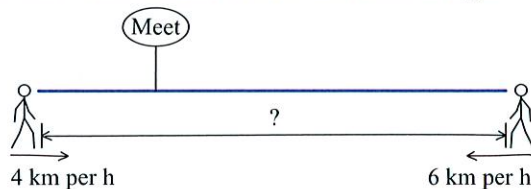
► A carpenter has a board 200 inches long and 12 inches wide. He makes 4 identical shelves and still has a piece of board 36 inches long left over. How long is each shelf? ◀

Dana tried to solve the problem as follows: She added 36 and 4, then scratched it out, and wrote 200×12 , but she thought that was too large so she scratched it out. Then she tried $2400 - 36$, which was also too large and discarded it. Then she calculated 4×36 and subtracted that from 200, getting 56. She then subtracted 12, and got 44.

Dana used a weak strategy called "Try all operations and choose." She obviously did not know what to do with this problem, although she was very good at solving one-step problems.

Do a quantitative analysis of this problem situation, and use it to make sense of the problem in a way that Dana did not. Use your analysis to solve the problem.

8. The problems listed below, and in Exercise 9, are from a Russian Grade 3 textbook.⁶ Solve the problems and compare their conditions and solutions.
- Two pedestrians simultaneously left two villages and walked toward each other, meeting after 3 hours. The first pedestrian walked 4 km in an hour, and the second walked 6 km in an hour. Find the distance between their villages.



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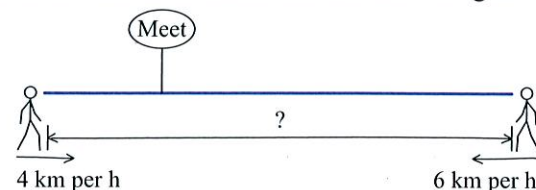
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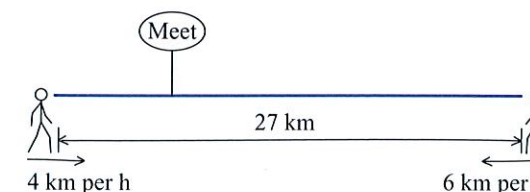
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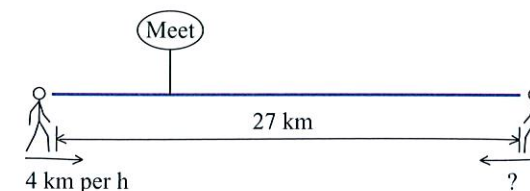


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- Two pedestrians simultaneously left two villages 27 km apart and walked toward each other. The first one walked 4 km per hour, and the second walked 6 km per hour. After how many hours did the pedestrians meet?



- Two pedestrians simultaneously left two villages 27 km apart and walked toward each other, meeting after 3 hours. The first pedestrian walked at a speed of 4 km per hour. At what speed did the second pedestrian walk?



- A 9. Two trains simultaneously left Moscow and Sverdlovsk, and traveled toward each other. The first traveled at 48 km per hour, and the second at 54 km per hour. How far apart were the two trains 12 hours after departure if it is 1822 km from Moscow to Sverdlovsk?

Supplementary Learning Exercises for Section 1.2

- We are comfortable in saying, for example, "She is prettier than she used to be." Explain why "prettiness" is difficult to quantify.
- Identify quantities suggested by the following situations, and make a drawing that could be useful in describing the situation.
 - Walt is taller than Sheila, and Tammi is shorter than Sheila.
 - Pedro makes \$3.75 more an hour than Jim does, but less than Kay.
 - I read 3 novels, with the first having 120 more pages than the second, and the second having 75 fewer pages than the third.
- The last part of the Tri-City Triathlon is a 10,000-meter run. When Les starts the run, she is 800 meters behind Pat. Les can run (on average) 250 meters per minute, but Pat can run (on average) only 225 meters per minute. Who wins, Les or Pat? Tell how far ahead the winner is when she crosses the finish line.
- What sort of drawing might help one solve these problems?
 - To make 6 popcorn balls, you need 8 cups of popcorn. How much popcorn would you need to make 24 popcorn balls?
 - A roll of film for an old camera cost \$4.80. You can take 12 pictures with such a roll. Then it cost 35¢ to get each picture developed. What was the cost of film and development for 12 pictures?
 - There were 270 students at the first game. That was 15 more students than attended the second game, and 18 more than the third game. What was the total attendance for the 3 games?
- Give two quantities that could be in mind for "That's an impressive building."
- Give two quantities that could be in mind for "Our basketball team played a good game."
- Towns P and Q are 500 miles apart. Train A leaves P, heading toward Q, at 50 miles per hour. On the next track, Train B leaves Q, heading toward P, at 45 miles per hour. (Hint: Drawing?)
 - How far apart are the locomotives of Trains A and B after 5 hours?
 - How far apart are the locomotives of the two trains after 8 hours?