Study Guide # 2

1. Double integrals; Midpoint Rule for rectangle : $\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x_i},\overline{y_j})\ \Delta A$;

2. Type I region $D : \begin{cases} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{cases}$; Type II region $D : \begin{cases} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{cases}$;

iterated integrals over Type I and II regions: $\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$ and $\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

3. Integral inequalities: $mA \le \iint_D f(x,y)\,dA \le MA$, where $A =$ area of D and $m \le f(x,y) \le M$ on D.

4. Change of Variables Formula in Polar Coordinates: if $D : \begin{cases} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{cases}$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\,\underset{\uparrow}{r}\,dr\,d\theta.$$

5. Applications of double integrals:

(a) Area of region D is $A(D) = \iint_D dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \ge 0$, is $V = \iint_D f(x,y)\,dA$

(c) Mass of D is $m = \iint_D \rho(x,y)\,dA$, where $\rho(x,y) =$ density (per unit area); sometimes write $m = \iint_D dm$, where $dm = \rho(x,y)\,dA$.

(d) Moment about the x-axis $M_x = \iint_D y\,\rho(x,y)\,dA$; moment about the y-axis $M_y = \iint_D x\,\rho(x,y)\,dA$.

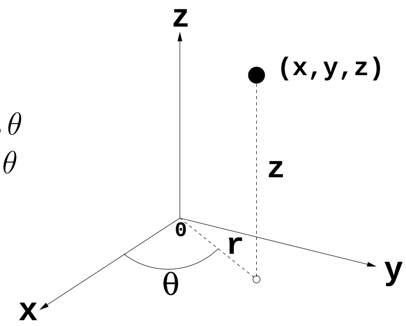
(e) Center of mass $(\overline{x},\,\overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$

<u>Remark</u>: centroid = center of mass when density is constant (this is useful).

6. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E: $\iiint_E f(x,y,z)\,dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x,y,z)\,dz\ dA$ for $E = \{(x,y) \in D,\ u(x,y) \le z \le v(x,y)\}$; volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x},\overline{y},\overline{z})$.

7. Cylindrical Coordinates (r, θ, z):

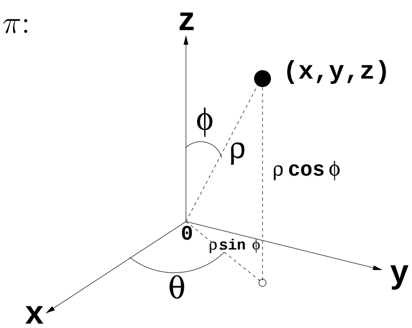
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

8. Spherical Coordinates (ρ, θ, ϕ), where $0 \le \phi \le \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

9. Triple integrals in Cylindrical Coordinates: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint_E f(x,y,z)\;dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$
$$\uparrow$$

10. Triple integrals in Spherical Coordinates: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2\sin\phi\;\;d\rho\,d\phi\,d\theta}$

$$\iiint_E f(x,y,z)\;dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$\uparrow$$

11. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$ and
$\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle$;
$\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

12. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t),\ \ y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),\, y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

Remarks:

(a) $\int_C f(x,y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$

(c) $\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$

13. Line integral of vector field $\vec{\mathbf{F}}(x,y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

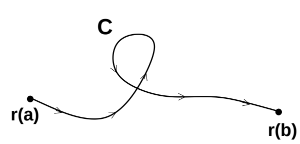
14. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}}\ \cdot\ d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

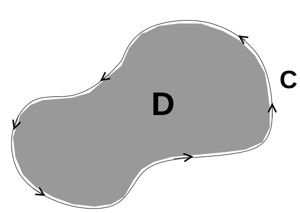
15. If $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$, then $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x,y)\, dx + Q(x,y)\, dy$; Work $= \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

16. <u>FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



17. A vector field $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.

18. <u>GREEN'S THEOREM</u>: $\int_C P(x,y)\, dx + Q(x,y)\, dy = \iint_D \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\right)\, dA$ ($C =$ boundary of D):



19. DEL OPERATOR: $\nabla = \frac{\partial}{\partial x}\,\vec{\mathbf{i}} + \frac{\partial}{\partial y}\,\vec{\mathbf{j}} + \frac{\partial}{\partial z}\,\vec{\mathbf{k}}$; if $\vec{\mathbf{F}}(x,y,z) = P(x,y,z)\,\vec{\mathbf{i}} + Q(x,y,z)\,\vec{\mathbf{j}} + R(x,y,z)\vec{\mathbf{k}}$, then

$$\text{curl } \vec{\mathbf{F}} = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div } \vec{\mathbf{F}} = \nabla \cdot \vec{\mathbf{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Properties of curl and divergence:

(i) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is a conservative vector field (i.e., $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$).

(ii) If curl $\vec{\mathbf{F}} = \vec{\mathbf{0}}$, then $\vec{\mathbf{F}}$ is *irrotational*; if div $\vec{\mathbf{F}} = 0$, then $\vec{\mathbf{F}}$ is *incompressible.*

(iii) *Laplace's Equation*: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.