7. Let $\boldsymbol{w1} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$ and $\boldsymbol{W} = \mathbf{span}\{\boldsymbol{w1}, \boldsymbol{w2}\}$.

a) Show that $\{\boldsymbol{w1}, \boldsymbol{w2}\}$ is an orthonormal set.

b) Use 10.4 to determine $\mathbf{proj}_{\boldsymbol{W}}\boldsymbol{u}$ where $\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

Section 10.3

The Gram – Schmidt Process

The ideas about projections in Section 10.2 actually tell us a way to construct an orthonormal basis from an existing basis provided we build the new basis one vector at a time.

The Gram-Schmidt process takes a basis $\mathbf{S} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_n}\}$ for a subspace of an inner product space $\mathbf{V}$ and produces a new basis $\mathbf{T} = \{\boldsymbol{w_1}, \boldsymbol{w_2}, ..., \boldsymbol{w_n}\}$ whose vectors form an orthonormal set. The process is often performed in two stages:

- First from the **S**-basis generate a basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$ of vectors that are mutually orthogonal. That is, $(\boldsymbol{v_i}, \boldsymbol{v_j}) = 0,\ i \neq j$.

- Second normalize each of the orthogonal basis vectors into a unit vector.

The first stage involves solving a set of equations and the second is easily performed using $\boldsymbol{w_i} = \boldsymbol{v_i} / \parallel \boldsymbol{v_i} \parallel$. At each step in the first stage we use projections onto subspaces.

The First Stage

Step 1. Define $\boldsymbol{v_1} = \boldsymbol{u_1}$.

Step 2. Look for a vector $\boldsymbol{v_2}$ in the $\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{u_2}\}$ that is orthogonal to $\boldsymbol{v_1}$. This will then guarantee that

$$\begin{aligned} \mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\} &= \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{u_2}\} && \text{since } \boldsymbol{v_1} = \boldsymbol{u_1} \\ &= \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} && \text{since } \boldsymbol{v_2} \text{ is a linear} \\ & && \text{combination of } \boldsymbol{v_1} \text{and } \boldsymbol{u_2} \end{aligned}$$

Let $\boldsymbol{v_2} = k_1\boldsymbol{v_1} + k_2\boldsymbol{u_2}$. Find k_1 and k_2 so that $(\boldsymbol{v1}, \boldsymbol{v2}) = \mathbf{0}$.

$$0 = (\boldsymbol{v_1}, \boldsymbol{v_2}) = k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) + k_2(\boldsymbol{v_1}, \boldsymbol{u_2})$$

We have one equation in two unknowns, so let $k_2 = 1$ and solve for k_1. We get

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}$$

thus we have

$$\boldsymbol{v_2} = \boldsymbol{u_2} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} = \boldsymbol{u_2} - \mathbf{proj}_{\boldsymbol{v_1}}\, \boldsymbol{u_2}$$

Step 3. Look for a vector $\boldsymbol{v_3}$ in $\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\}$ that is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. This will guarantee that $\mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$. Let $\boldsymbol{v_3} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + k_3\boldsymbol{u_3}$. Find k_1, k_2, and k_3 so that $(\boldsymbol{v_1}, \boldsymbol{v_3}) = \mathbf{0}$ and $(\boldsymbol{v_2}, \boldsymbol{v_3}) = 0$.

$$\begin{aligned} 0 &= (\boldsymbol{v_1}, \boldsymbol{v_3}) = \boldsymbol{k_1}(\boldsymbol{v_1}, \boldsymbol{v_1}) + \boldsymbol{k_2}(\boldsymbol{v_1}, \boldsymbol{v_2}) + \boldsymbol{k_3}(\boldsymbol{v_1}, \boldsymbol{u_3}) \\ 0 &= (\boldsymbol{v_2}, \boldsymbol{v_3}) = \boldsymbol{k_1}(\boldsymbol{v_2}, \boldsymbol{v_1}) + \boldsymbol{k_2}(\boldsymbol{v_2}, \boldsymbol{v_2}) + \boldsymbol{k_3}(\boldsymbol{v_2}, \boldsymbol{u_3}) \end{aligned}$$

Since by construction $(\boldsymbol{v1}, \boldsymbol{v2}) = 0$ the preceding equations simplify to

$$\begin{array}{lll} k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) & & + k_3(\boldsymbol{v_1}, \boldsymbol{u_3}) = 0 \\ & k_2(\boldsymbol{v_2}, \boldsymbol{v_2}) & + k_3(\boldsymbol{v_2}, \boldsymbol{u_3}) = 0 \end{array}$$

Thus we have 2 equations in 3 unknowns. Let $k_3 = 1$, then we find that

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})} \qquad \text{and} \qquad k_2 = \frac{-(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}$$

and hence

$$\boldsymbol{v_3} = \boldsymbol{u_3} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} - \frac{(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}\boldsymbol{v_2} = \boldsymbol{u_3} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}}\, \boldsymbol{u_3}$$

Other steps: $\boldsymbol{v_k} = \boldsymbol{u_k} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_{k-1}}\}}\, \boldsymbol{u_k}$

The Second Stage

The orthonormal basis for $\mathbf{V}$ is given by

$$\{\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_n}\} = \left\{ \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|}, \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|}, \ldots, \frac{\boldsymbol{v_n}}{\|\boldsymbol{v_n}\|} \right\}$$

Example 1. Let **V**= **span**$\{u_1, u_2, u_3\}$ where

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Use the Gram-Schmidt process to find an orthonormal basis for **V**.

Step 1. Define $v_1 = u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}.$

Step 2. Compute $v_2 = u_2 - \mathbf{proj}_{v_1}\, u_2 = u_2 - \frac{(v_1,u_2)}{(v_1,v_1)} v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{14}{21} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}.$

Step 3. Compute $v_3 = u_3 - \mathbf{proj}_{\mathbf{span}\{v_1, v_2\}}\, u_3 = u_3 - \frac{(v_1,u_3)}{(v_1,v_1)} v_1 - \frac{(v_2,u_3)}{(v_2,v_2)} v_2 =$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{21} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} - \frac{(-2/3)}{15/9} \begin{bmatrix} \frac{-1}{3} \\ \frac{-2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{17}{35} \\ \frac{54}{35} \\ \frac{7}{5} \\ \frac{-22}{35} \end{bmatrix}.$$

The set $\{v_1, v_2, v_3\}$ is an orthogonal basis for **V**. An orthonormal basis is obtained by dividing each vector by it length.

$$w_1 = \frac{v_1}{\sqrt{21}}, \qquad w_2 = \frac{v_2}{\sqrt{17/9}}, \qquad w_3 = \frac{v_3}{\sqrt{6090/1225}}$$

For $\mathbf{V} = R^n$ and the standard inner product both stages of the Gram-Schmidt process are available in MATLAB routine **gschmidt**. Type **help gschmidt** for more details. The following examples illustrate the use of routine **gschmidt**.

Example 2. Let $\mathbf{S} = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis for R^3. To find an orthonormal basis from **S** using MATLAB enter the vectors u_1, u_2, u_3 as columns of a matrix A and type

B = gschmidt(A)

The display generated is

```
B =

    0.4472    0.7807   -0.4364
         0    0.4880    0.8729
    0.8944   -0.3904    0.2182
```

The columns of B are an orthonormal basis for R^3.

Example 3. We will show how to find an orthonormal basis for R^4 containing scalar multiples of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

First enter v_1 and v_2 into MATLAB as vectors **v1** and **v2**, respectively. To find a basis containing scalar multiples of v_1 and v_2, use commands

A = [v1 v2 eye(4)]
rref(A)

The display indicates that the first four columns of A form a basis for R^4. The command **S = A(:,1:4)** produces the matrix with those columns. Type the command

T = gschmidt(S)

The display is

```
T =
    0.5774   -0.3780    0.7237         0
         0    0.3780    0.1974    0.9045
    0.5774    0.7559   -0.0658   -0.3015
   -0.5774    0.3780    0.6580   -0.3015
```

Column 1 of T is $\left(\frac{1}{\|v_1\|}\right) v_1$ and column 2 of T is $\left(\frac{1}{\|v_2\|}\right) v_2$, hence the columns of T form the desired orthonormal basis for R^4.

Explain what to do if **rref(A)** did not indicate that the first four columns of A form a basis for R^4.