

MA 22000 Lesson 20 Notes
 (Calculus part of text) Section 4.4
 Derivatives of Exponential Functions

Deriving the formula for a derivative of the function $f(x) = e^x$:

$$\begin{aligned} \frac{d(e^x)}{dx} &= \lim_{h \rightarrow 0} \left(\frac{e^{x+h} - e^x}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{e^x e^h - e^x}{h} \right) \\ &= \lim_{h \rightarrow 0} \left(\frac{e^x (e^h - 1)}{h} \right) \\ &= \lim_{h \rightarrow 0} \left[e^x \left(\frac{e^h - 1}{h} \right) \right] \end{aligned}$$

For h values very close to zero, the value of $\left(\frac{e^h - 1}{h} \right)$ is approximately 1. (You can see a table on page 228 of your textbook to verify this.) Therefore:

$$\begin{aligned} \lim_{h \rightarrow 0} \left[e^x \left(\frac{e^h - 1}{h} \right) \right] &= \lim_{h \rightarrow 0} (e^x \cdot 1) \\ \lim_{h \rightarrow 0} \left[e^x \left(\frac{e^h - 1}{h} \right) \right] &= e^x \end{aligned}$$

This argument leads to the following conclusion:

Derivative of e^x : $\frac{d}{dx}(e^x) = e^x$

**To find the rule to determine the derivative of exponential functions with bases other than e , we use the following fact. By the ‘composition of inverse functions’ in algebra, $e^{\ln a} = a$. Examine the following.

$e^{\ln a} = a$ by the definition of inverse functions

$$\begin{aligned} e^{(\ln a)x} &= \left(e^{\ln a} \right)^x && \text{Product rule of exponents} \\ &= a^x \end{aligned}$$

Therefore: $a^x = e^{(\ln a)x}$

Use this fact to find the derivative of a^x .

Derivative of a^x :

$$\begin{aligned}\frac{d}{dx}(a^x) &= \frac{d}{dx}(e^{(\ln a)x}) \\ &= (e^{(\ln a)x})(\ln a) \\ &= (\ln a)a^x\end{aligned}$$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

****Note: There are no homework problems assigned with exponential functions with any base other than e . Therefore the above derivative rule is not necessary for students. I simply provide it for information purposes only.**

Remember the 'chain rule' must be used if the exponent is an expression (a function) other than the variable.

Derivatives of exponential functions (base e) using the chain rule:

$$\frac{d}{dx}(e^{g(x)}) = (e^{g(x)})(g'(x))$$

Ex 1: Find the derivatives:

a) $y = e^{3x}$

b) $y = e^{(5x+6x^2)}$

c) $D_x(3.8e^{1.5x})$

d) $h(x) = -12e^{x^2}$

$$e) \quad y = 4e^{4x^2+2x}$$

$$f) \quad y = 3x^3 e^{-4x}$$

$$g) \quad g(x) = (5x^2 - 7x^3)e^{-3x}$$

$$h) \quad y = \frac{e^{2x}}{3x^2 + 5}$$

i) $f(x) = (e^{4x^2} + 10x)^4$

j) $y = \frac{200}{8 + 3e^x}$

Ex 2:

The quantity (in grams) of a radioactive substance present after t years is given by the model

$$Q(t) = 100e^{-0.421t}.$$

- a) Find the quantity when $t = 0$.
- b) Find the rate of change after (a) 2 years, (b) 5 years.

Ex 3:

The growth of the world population (in millions) can be approximated by the function $A(t) = 3100e^{(0.0166t)}$ where t is the number of years since 1960. Find the instantaneous rate of change in the world population for the year (a) 2010 and (b) 2015.

Ex 4:

The amount (in grams) of a sample of an element present after t years is given by $A(t) = 400e^{-0.4t}$. Find the rate of change of the quantity after (a) 3 years, (b) 5 years, and (c) 15 years.