MA 22000 Lesson 21 Notes

How would a person solve an equation with a variable in an exponent, such as $2^{(x-5)} = 9$? (We cannot re-write this equation easily with the same base.) A notation was developed so that equations such as this one could be solved. This 'new' function is called a logarithm or a logarithmic function.

Definition of a logarithm or logarithmic function:
Let $a > 0,\ a \neq 1,$ and $x > 0$ (a any positive number other than 1 and x any positive number), then the logarithm y and the logarithmic function is defined as
$f(x) = y = \log_a x$ and is equivalent to $a^y = x$.

****It is important to understand that a logarithm is an <u>exponent</u>!****

<u>Ex 1</u>: Write each exponential expression as a logarithmic expression. (Convert from exponential form to logarithmic form.)

$a)$ $4^2 = 16$ $\quad$ $b)$ $m^{-5} = q$ $\quad$ $c)$ $\left(\frac{2}{3}\right)^n = \frac{8}{27}$

$d)$ $10^5 = a$ $\quad$ $e)$ $e^{(x-2)} = 8$ $\quad$ $f)$ $n^{12} = 5000$

<u>Ex 2</u>: Write each logarithmic expression as an exponential expression. (Convert from logarithmic form to exponential form.)

$a)$ $\log_5 125 = 3$ $\quad$ $b)$ $\log_r 100 = 3$ $\quad$ $c)$ $\log_\pi n = -3$

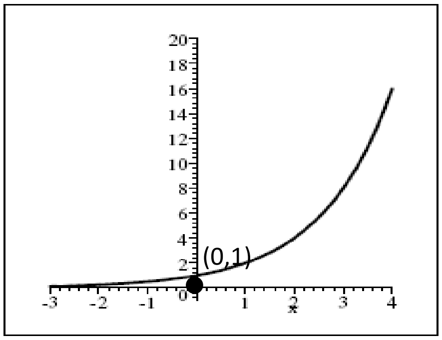
$d)$ $\log_{\left(\frac{1}{4}\right)} 9 = m$ $\quad$ $e)$ $\log_b 25 = (q+1)$ $\quad$ $f)$ $\log_{0.2} n = -4$

<u>Ex 3</u>: Find each logarithm. (Remember, a logarithm is an exponent! You are asked to find an exponent, if it exists.)

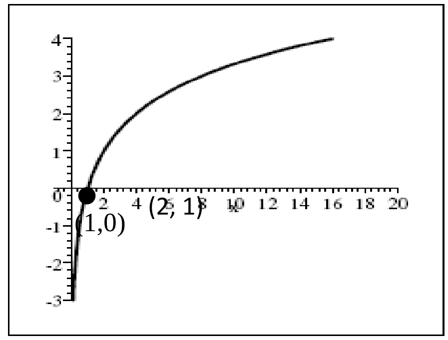
$a)$ $\log_2 16 =$ $\quad$ $b)$ $\log_5 \left(\frac{1}{5}\right) =$ $\quad$ $c)$ $\log_{10} 10000 =$

$d)$ $\log_3 (-81) =$ $\quad$ $e)$ $\log_4 64 =$ $\quad$ $f)$ $\log_{(1/2)} 8 =$

Below is a graph of $y = 2^x$ and its inverse, $x = 2^y$ *or* $y = \log_2 x$.



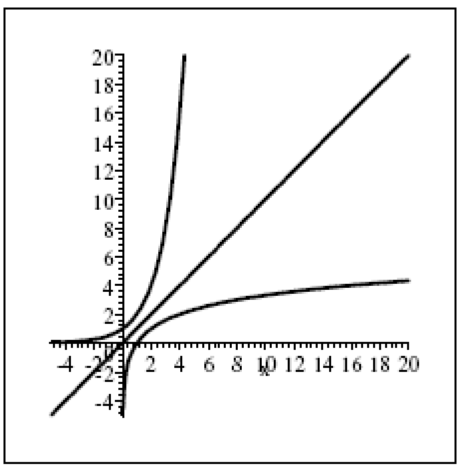
$y = 2^x$



$x = 2^y$

If you imagine the line *y = x*, you can see the symmetry about that line. A function and its inverse will have symmetry about the line *y = x*.

Below are both graphs on the same coordinate system along with the line *y = x*.



There is another graph of $f(x) = y = 2^x$ and the inverse, $f^{-1}(x) = \log_2 x$ on page 91 (calculus part of textbook). Again, you can see the symmetry about the line *y = x*.

PROPERTIES OF LOGARITHMS: Let *x* and *y* be any positive real numbers and *r* be any real number. Let *a* be a positive real number other than 1 ($a \neq 1$). Then the following properties exist.

a. $\log_a xy = \log_a x + \log_a y$

b. $\log_a \frac{x}{y} = \log_a x - \log_a y$

c. $\log_a x^r = r \log_a x$

d. $\log_a a = 1$

e. $\log_a 1 = 0$

f. $\log_a a^r = r$

The properties of logarithms are easy to prove, if you remember that a logarithm is an exponent. Logarithms 'behave' like exponents. When multiplying, exponents are added (property *a*). When dividing, exponents are subtracted (property *b*). When a power is raised to another power, the exponents are multiplied (property *c*). Any real number to the first power is itself (property *d*). Any real number to the zero power is 1 (property *e*). To prove property *f*, just put the logarithmic expression in exponential form.

<u>Ex 4</u>: Use the properties of logarithms to write the expression as a sum, difference, or product of simpler logarithms.

$a)\quad \log_2(5\sqrt{x})$

$b)\quad \log_b\left(\dfrac{2x}{yz^2}\right)$

$c)\quad \log_5\left(\dfrac{1}{625x^3y^2}\right)$

<u>Ex 5</u>: Suppose $\log_a 3 = m,\ \log_a 4 = n,$ and $\log_a 5 = r$. Use the properties of logarithms to find the following.

$a)\quad \log_a 12$

$b)\quad \log_a \dfrac{25}{3}$

$c)\quad \log_a 48a^2$

Your scientific 1-line calculator will find logarithms using base 10 (**common logarithms**) or base e (**natural logarithms**). If a logarithm is written $\log x$ (with no base indicated), it is assumed to be a common logarithm, or base 10 logarithm. If a logarithm is written $\ln x$, it is assumed to be a natural logarithm (base e logarithm). Use your calculator to approximate each of the following to 4 decimal places. (Enter the number, then either the log key or the ln key.)

$a)\quad \ln 22$ $\qquad b)\quad \log 49$

$c)\quad \ln 0.052$ $\qquad d)\quad \log 3.2$

Change of base Theorem for Logarithms:
If x is any positive number and if a and b are positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Most often, base b is chosen to be 10 or e, if a calculator will be used to approximate.

$$\log_a x = \frac{\log x}{\log a} \text{ or } \log_a x = \frac{\ln x}{\ln a}$$

<u>Ex 6</u>: Use natural logarithms to evaluate the logarithm. Give an exact answer and an approximation to the nearest thousandth.

$a)\quad \log_4 20$ $\qquad b)\quad \log_{\left(\frac{1}{3}\right)} 0.5$

$c)\quad \log_{19}(0.03)$ $\qquad d)\quad \log_{1.2} 5$

Solving some simple logarithmic equations:

Ex 7: Solve each equation. Hint: Convert to exponential form. Remember you can only find logarithms of positive numbers, so check your answers.

$a)\quad \log_x 64 = 6$

$b)\quad \log_x 27 = -3$

$c)\quad \log_8\left(\frac{1}{64}\right) = x$

$d)\quad \log_3(2x-5) = 2$

$e)\quad \log(x+5) + \log(x+2) = 1$

Solving some simple exponential equations:

Ex 8: Solve each equation by using natural logarithms (take the natural log of both sides). Approximate to four decimal points, if needed.

$a)\quad 6^x = 15$

$b)\quad e^{k-2} = 4$

$c)\quad 4^{2x+3} = 6^{x-1}$

Applied problems:

<u>Ex 9:</u> Leigh plans to invest $1000 into an account. Find the interest rate that is needed for the money to grow to $1500 in 8 years if the interest is compounded continuously.

<u>Ex 10:</u> The magnitude of an earthquake, measured on the Richter scale, is given by $R(I)=\log\left(\frac{I}{I_0}\right)$ where I is the amplitude registered on a seismograph located 100 km from the epicenter of the earthquake and I_0 is the amplitude of a certain small size earthquake. Find the Richter scale rating of a earthquake with the following amplitude. (Round to the nearest tenth.)

$25000I_0$