

MA 22000 Lesson 25 Notes
Section 5.3 (2nd half of textbook)

Higher Derivatives:

In this lesson, we will find a derivative of a derivative. A second derivative is a derivative of the first derivative. A third derivative is a derivative of a second derivative, etc.

Notation for Higher Derivatives

For the second derivative of a function, $y = f(x)$, any of the following notations may be used.

$$f''(x), \frac{d^2y}{dx^2}, D_x^2[f(x)], y''$$

For the third derivative of a function, similar notation is used.

$$f'''(x), \frac{d^3y}{dx^3}, y''', D_x^3[f(x)]$$

For $n > 4$, the n th derivative is written

$$f^{(n)}(x), y^{(n)}(x), \frac{d^n y}{dx^n}, \text{ or } \frac{d^n}{dx^n}[f(x)], .$$

(See the caution in the middle of page 275 of the text.)

Ex 1: (a) If $f(x) = 3x^3 - 4x^2 + 5x - 8$; find the 1st, 2nd, 3rd, and 4th derivatives. (b) Find $f''(-2)$ and $f'''(1000)$

a)

$$f'(x) = 9x^2 - 8x + 5$$

$$f''(x) = 18x - 8$$

$$f'''(x) = 18$$

$$f^{(4)} = 0$$

b)

$$f''(-2) = 18(-2) - 8$$

$$= -36 - 8$$

$$= -44$$

$$f'''(1000) = 18$$

Ex 2: Find the first and second derivatives of $y = 2(x^3 - 2)^4$.

Ex 3: Find the first three derivatives of $g(x) = 2x^5 - \frac{3}{x^2}$.

Ex 4: Find the first and second derivatives of each.

a) $y = \frac{x+4}{x-3}$

b) $f(n) = \frac{-5}{(n+3)^2}$

Ex 5: Find the second derivative two different ways; (1) using the product rule, and (2) using the basic rules after finding the product of the factors.

$$g(x) = 2x^3(x^2 - 3x + 2)$$

Ex 6: Find the second derivative of each.

a) $f(x) = 2x(e^x)$

b) $g(x) = \frac{\ln x}{2x}$

Remember a derivative is a rate of change. In the real world, the rate of change of a distance (position of a vehicle along a straight line) is **velocity**. ($v(t) = s'(t)$) The instantaneous rate of change of velocity is called **acceleration**. $a(t) = \frac{d}{dt} v(t) = v'(t) = s''(t)$

This is paragraph from page 276 of the textbook.

- (1) If the velocity is positive and the acceleration is positive, the velocity is increasing, so the vehicle is speeding up.
- (2) If the velocity is positive and the acceleration is negative, the vehicle is slowing down.
- (3) A negative velocity and a positive acceleration mean the vehicle is backing up and slowing down.
- (4) If both the velocity and acceleration are negative, the vehicle is speeding up in the negative direction.

In figure 36 below, the graph is concave upward in the interval from $x = -\infty$ to $x = 1$, illustrated by the interval $(-\infty, 1)$. The graph is also concave upward from $x = 3$ to infinity, illustrated by the interval $(3, \infty)$. The graph is concave downward in the interval $(1, 3)$. Notice: The intervals of concavity are given from one x value to another x value.

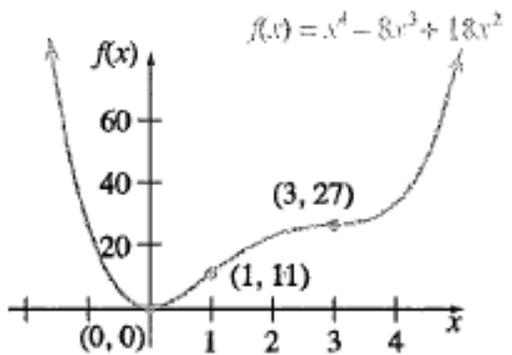


FIGURE 36

The point where a graph (function) changes from concave one way to concave the other way is called an **inflection point or point of inflection**. The graph in figure 36 above (and on page 280 of the text) has two inflection points, $(1, 11)$ and $(3, 27)$.

****One other point about concavity****

Remember that the first derivative is the slope of a tangent line to a curve. Suppose the tangent lines to a curve are drawn as x gets larger. What happens to the slopes of the tangent lines?

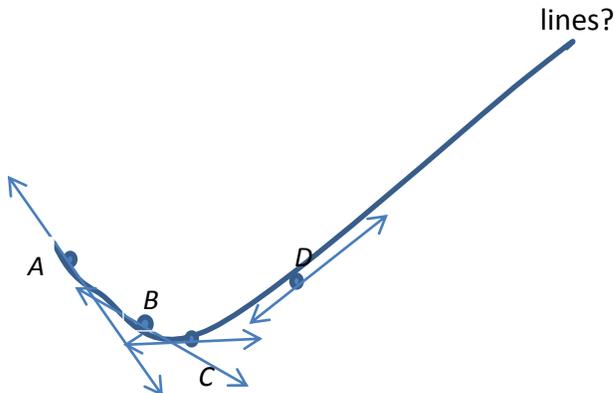


FIGURE 1

The slope of the tangent line at point A is negative. At point B (which is greater than A), the slope is negative, but a larger numeric value. At point C , the slope is almost 0 (larger than the slopes at points A and B). At point D the slope of the tangent line is positive. Conclusion: As x gets larger (A to B to C to D , left to right), the slopes of the tangent lines are getting larger. The change in the derivative is increasing.

The curve in figure 1 is **concave upward**. Notice the curve lies above its tangent lines.

Suppose the tangent lines to the curve below are drawn as x gets larger.

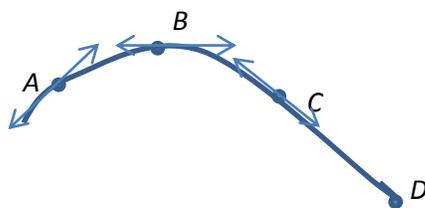


FIGURE 2

The slope of the tangent line at point A is a positive number. At point B the slope is still positive, but much smaller (almost zero). By point C , the slope is negative and at point D , the slope of the tangent line is an even smaller negative number. Conclusion: As x gets larger (A to B to C to D , left to right), the slopes of the tangent lines are getting smaller. The change in the derivative is decreasing.

The curve shown in figure 2 is **concave downward**. Notice the curve lies below its tangent lines.

Test for Concavity.

If f is a function with derivatives f' and f'' existing at all points in an interval (a, b) . Then f is concave upward on (a, b) if $f'' > 0$ for all x in the interval and concave downward on (a, b) if $f'' < 0$ for all x in the interval.

Ex 8: Use the test for concavity and a **2nd derivative sign chart** to find where the function below is concave upward or concave downward.

$$g(x) = x^4 - 8x^3 + 18x^2$$

$$g'(x) = 4x^3 - 24x^2 + 36x$$

$$g''(x) = 12x^2 - 48x + 36$$

$$12x^2 - 48x + 36 = 0 \quad \text{Note: These values are not critical values.}$$

$$12(x^2 - 4x + 3) = 0 \quad \text{They are simply values where the 2nd derivative is zero.}$$

$$12(x-3)(x-1) = 0$$

$$x-3=0 \quad x-1=0$$

$$x=3 \quad x=1 \quad 12$$

$$x-3$$

$$x-1$$

Result

$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
+	+	+

Answer: Concave upward _____

Concave downward _____

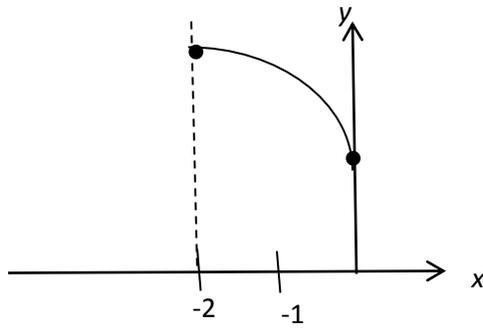
List any inflection points: _____

Ex 9: $f(x) = -2x^3 + 3x^2 + 72x$

- Find critical values.
- Find intervals of increasing or decreasing.
- Find any relative extrema.
- Find intervals of concavity.
- Find any inflection points.

Example 10: Look at each graph of a function f . Determine the signs of $f'(x)$ and $f''(x)$ for the interval of each graph shown.

a)



b)

