

Study Guide for Exam # 2

1. Relative/local extrema; critical points ( $\nabla f = \vec{0}$  or  $\nabla f$  does not exist);  $2^{nd}$  Derivatives Test: A critical points is a local min if  $D = f_{xx}f_{yy} - f_{xy}^2 > 0$  and  $f_{xx} > 0$ , local max if  $D > 0$  and  $f_{xx} < 0$ , saddle if  $D < 0$ ; absolute extrema; Max-Min Problems; **Lagrange Multipliers:** *Extremize*  $f(\vec{x})$  subject to a constraint  $g(\vec{x}) = C$ , solve the system:  $\nabla f = \lambda \nabla g$  and  $g(\vec{x}) = C$ .

2. Double integrals; **Midpoint** Rule for rectangle :  $\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$ ;

3. Type I region  $D : \begin{cases} g_1(x) \leq y \leq g_2(x) \\ a \leq x \leq b \end{cases}$  ; Type II region  $D : \begin{cases} h_1(y) \leq x \leq h_2(y) \\ c \leq y \leq d \end{cases}$  ;

iterated integrals over Type I and II regions:  $\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$  and

$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$ , respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities:  $m A \leq \iint_D f(x, y) dA \leq M A$ , where  $A = \text{area of } D$  and  $m \leq f(x, y) \leq M$  on  $D$ .

5. Change of Variables Formula in Polar Coordinates: if  $D : \begin{cases} h_1(\theta) \leq r \leq h_2(\theta) \\ \alpha \leq \theta \leq \beta \end{cases}$ , then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

↑

6. Applications of double integrals:

(a) Area of region  $D$  is  $A(D) = \iint_D dA$

(b) Volume of solid under graph of  $z = f(x, y)$ , where  $f(x, y) \geq 0$ , is  $V = \iint_D f(x, y) dA$

(c) Mass of  $D$  is  $m = \iint_D \rho(x, y) dA$ , where  $\rho(x, y) = \text{density (per unit area)}$ ; sometimes write  $m = \iint_D dm$ , where  $dm = \rho(x, y) dA$ .

(d) Moment about the  $x$ -axis  $M_x = \iint_D y \rho(x, y) dA$ ; moment about the  $y$ -axis  $M_y = \iint_D x \rho(x, y) dA$ .

(e) Center of mass  $(\bar{x}, \bar{y})$ , where  $\bar{x} = \frac{M_y}{m} = \frac{\iint_D x \rho(x, y) dA}{\iint_D \rho(x, y) dA}$ ,  $\bar{y} = \frac{M_x}{m} = \frac{\iint_D y \rho(x, y) dA}{\iint_D \rho(x, y) dA}$

Remark: centroid = center of mass when density is constant (this is useful).

(f) The area of the surface  $S$  with equation  $z = f(x, y)$ ,  $(x, y) \in D$ , where  $f_x$  and  $f_y$  are continuous, is

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1} dA.$$

7. Elementary solids  $E \subset \mathbb{R}^3$  of Type 1, Type 2, Type 3; triple integrals over solids  $E$ :

$$\iiint_E f(x, y, z) dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x, y, z) dz dA \text{ for } E = \{(x, y) \in D, u(x, y) \leq z \leq v(x, y)\};$$

volume of solid  $E$  is  $V(E) = \iiint_E dV$ ; applications of triple integrals, mass of a solid, moments about the coordinate planes  $M_{xy}$ ,  $M_{xz}$ ,  $M_{yz}$ , center of mass of a solid  $(\bar{x}, \bar{y}, \bar{z})$ .

8. **Cylindrical Coordinates (CC)**  $(r, \theta, z)$ :

$$\text{From CC to RC : } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

Going from RC to CC use  $x^2 + y^2 = r^2$  and  $\tan \theta = \frac{y}{x}$  (make sure  $\theta$  is in correct quadrant).

9. **Spherical Coordinates (SC)**  $(\rho, \theta, \phi)$ , where  $0 \leq \phi \leq \pi$ :

$$\text{From SC to RC : } \begin{cases} x = (\rho \sin \phi) \cos \theta \\ y = (\rho \sin \phi) \sin \theta \\ z = \rho \cos \phi \end{cases}$$

Going from RC to SC use  $\rho = \sqrt{x^2 + y^2 + z^2}$ ,  $\tan \theta = \frac{y}{x}$  and  $\cos \phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$ .

10. Triple integrals in Cylindrical Coordinates (CC):  $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}, \quad dV = r dz dr d\theta$

$$\iiint_E f(x, y, z) dV = \iiint_E f(r \cos \theta, r \sin \theta, z) \underbrace{r}_{\uparrow} dz dr d\theta$$

11. Triple integrals in Spherical Coordinates:  $\begin{cases} x = (\rho \sin \phi) \cos \theta \\ y = (\rho \sin \phi) \sin \theta \\ z = \rho \cos \phi \end{cases}, \quad dV = \rho^2 \sin \phi d\rho d\phi d\theta$

$$\iiint_E f(x, y, z) dV = \iiint_E f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \underbrace{\rho^2 \sin \phi}_{\uparrow} d\rho d\phi d\theta$$

Recall that

$$\cos(\pi/4) = \sin(\pi/4) = \frac{\sqrt{2}}{2}, \quad \cos(\pi/6) = \sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad \cos(\pi/3) = \sin(\pi/6) = \frac{1}{2}.$$

12. Vector fields on  $\mathbb{R}^2$  and  $\mathbb{R}^3$ :  $\tilde{\mathbf{F}}(x, y) = \langle P(x, y), Q(x, y) \rangle$  and  $\tilde{\mathbf{F}}(x, y, z) = \langle P(x, y), Q(x, y), R(x, y) \rangle$ ;  $\tilde{\mathbf{F}}$  is a conservative vector field if  $\tilde{\mathbf{F}} = \nabla f$ , for some real-valued function  $f$ .