Study Guide # 3

You also need Study Guides # 1 and # 2 for the Final Exam

1. Line integral of a function $f(x,y,z)$ along C, parameterized by $x = x(t), \quad y = y(t), \quad z = z(t)$ and $a \le t \le b$, is

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t),\, y(t),\, z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\int_C f(x,y,z)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\int_C f(x,y,z)\, dx = \int_a^b f(x(t),y(t),z(t))\, x'(t)\, dt$

(c) $\int_C f(x,y,z)\, dy = \int_a^b f(x(t),y(t),z(t))\, y'(t)\, dt$

(d) $\int_C f(x,y,z)\, dy = \int_a^b f(x(t),y(t),z(t))\, z'(t)\, dt$

2. Line integral of vector field $\mathbf{F}(x,y,z)$ along C, parameterized by $\mathbf{r}(t)$ and $a \le t \le b$, is given by

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

3. Connection between line integral of vector fields and line integral of functions:

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (\mathbf{F} \bullet \mathbf{T})\, ds$$

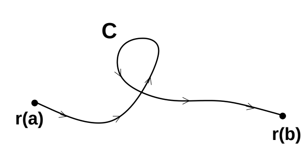
where $\mathbf{T}$ is the unit tangent vector to the curve C.

4. If $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$, then

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C P(x,y,z)\, dx + Q(x,y,z)\, dy + R(x,y,z)\, dz\,;$$

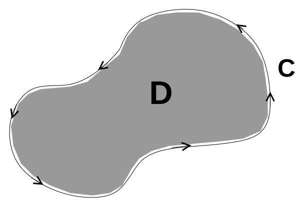
Work $= \int_C \mathbf{F} \bullet d\mathbf{r}$.

5. <u>Fundamental Theorem of Calculus for Line Integrals</u>: $\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$:



6. A vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is *conservative* (i.e. $\mathbf{F} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$. If $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$, then $f_x = P$ and $f_y = Q$. Start out with integrating one of the equation and use the other to get the constant. Similarly, a vector field $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ is *conservative* (i.e. $\mathbf{F} = \nabla f$) if $\dfrac{\partial^2 P}{\partial y \partial z} = \dfrac{\partial^2 Q}{\partial x \partial z} = \dfrac{\partial^2 R}{\partial x \partial y}$.

7. <u>Green's Theorem</u>: $\displaystyle\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$ ($C =$ boundary of D):



As a consequence of Green's Theorem one has

$$\frac{1}{2}\int_C x\,dy - y\,dx = \int_C x\,dy = -\int_C y\,dx = Area(D)$$

8. Del Operator: $\dfrac{\partial}{\partial x}\,\mathbf{i} + \dfrac{\partial}{\partial y}\,\mathbf{j} + \dfrac{\partial}{\partial z}\,\mathbf{k}$; if $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\mathbf{k}$, then

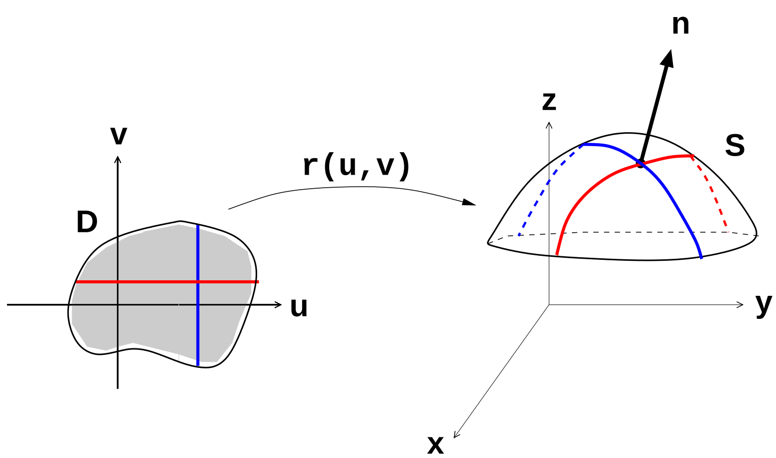
$$\text{curl }\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \quad \text{and} \quad \text{div }\mathbf{F} = \nabla \bullet \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note that $\text{curl}(\nabla f) = 0$. Some properties of curl and divergence:

(i) If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field (i.e., $\mathbf{F}(\mathbf{x}) = \nabla f(\mathbf{x})$).

(ii) If curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is *irrotational*; if div $\mathbf{F} = 0$, then $\mathbf{F}$ is *incompressible*.

9. Parametric surface S: $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, where $(u,v) \in D$:



Normal vector to surface S : $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$; tangent planes and normal lines to parametric surfaces.

10. Surface area of a surface S:

(i) $A(S) = \iint_D dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA$

(ii) If S is the graph of $z = h(x, y)$ above D, then $A(S) = \iint_D \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, dA.$

(iii) If S is the surface of revolution obtained from $y = f(x)$ revolving about the x-axis, $a \leq x \leq b$, then

$$\mathbf{r}(x, \theta) = \langle x, \ f(x)\cos\theta, \ f(x)\sin\theta \rangle, \qquad D = \{(x, \theta) \mid a \leq x \leq b, \ \ 0 \leq \theta \leq 2\pi\}.$$

Then

$$|\mathbf{r}_x \times \mathbf{r}_\theta| = f(x)\sqrt{1 + [f'(x)]^2}$$

Then the area of S becomes:

$$A(s) = \int_0^{2\pi} \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx \, d\theta = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} \, dx.$$

Remark: $dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dA =$ differential of surface area; while $d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) \, dA$

11. The surface integral of $f(x, y, z)$ over the surface S:

(i) $\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$

(ii) If S is the graph of $z = h(x, y)$ above D, then

$$\iint_S f(x, y, z) \, dS = \iint_D f(x, y, h(x, y)) \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} \, dA.$$

12. Recall,

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}, \qquad d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \, (|\mathbf{r}_u \times \mathbf{r}_v| \, dA) = \mathbf{n} \, dS.$$

The surface integral of $\mathbf{F}$ over the surface S

$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \iint_S (\mathbf{F} \bullet \mathbf{n}) \, dS$$

If S is the graph of $z = h(x, y)$ above D, with $\mathbf{n}$ oriented upward, and $\mathbf{F} = \langle P, \ Q, \ R \rangle$, then

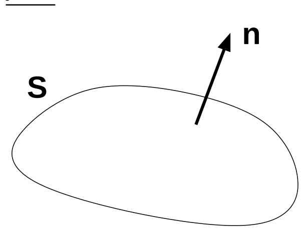
$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_D \left(-P \frac{\partial h}{\partial x} - Q \frac{\partial h}{\partial y} + R \right) dA.$$

(i) Connection between surface integral of a vector field and a function:

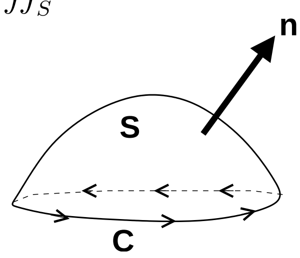
$$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S (\mathbf{F} \bullet \mathbf{n}) \, dS.$$

The above gives another way to compute $\iint_S \mathbf{F} \bullet d\mathbf{S}$.

(ii) $\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S (\mathbf{F} \bullet \mathbf{n}) \, dS =$ *flux* of $\mathbf{F}$ across the surface S.



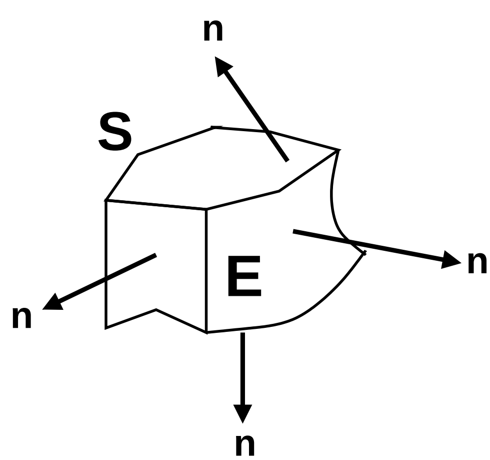
13. <u>Stokes' Theorem</u>: $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \bullet d\mathbf{S}$ (recall, $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$).



$\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r} =$ *circulation* of $\mathbf{F}$ around C.

14. <u>The Divergence Theorem/Gauss' Theorem</u>: $\displaystyle\iint_S \mathbf{F} \bullet d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \; dV$

(recall, $\operatorname{div} \mathbf{F} = \nabla \bullet \mathbf{F}$).



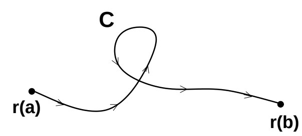
15. Summary of Line Integrals and Surface Integrals:

LINE INTEGRALS	SURFACE INTEGRALS
$C : \mathbf{r}(t)$, where $a \le t \le b$	$S : \mathbf{r}(u,v)$, where $(u,v) \in D$
$ds = \lvert\mathbf{r}'(t)\rvert\, dt =$ differential of arc length	$dS = \lvert\mathbf{r}_u \times \mathbf{r}_v\rvert\, dA =$ differential of surface area
$\int_C ds =$ length of C	$\iint_S dS =$ surface area of S
$\int_C f(x,y,z)\, ds = \int_a^b f(\mathbf{r}(t))\, \lvert\mathbf{r}'(t)\rvert\, dt$ (independent of orientation of C)	$\iint_S f(x,y,z)\, dS = \iint_D f(\mathbf{r}(u,v))\, \lvert\mathbf{r}_u \times \mathbf{r}_v\rvert\, dA$ (independent of normal vector $\mathbf{n}$)
$d\mathbf{r} = \mathbf{r}'(t)\, dt$	$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\ dA$
$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt$ (depends on orientation of C)	$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \bullet (\mathbf{r}_u \times \mathbf{r}_v)\ dA$ depends on normal vector $\mathbf{n} = \dfrac{\mathbf{r}_u \times \mathbf{r}_v}{\lvert\mathbf{r}_u \times \mathbf{r}_v\rvert}$
$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (\mathbf{F} \bullet \mathbf{T})\ ds$ The *circulation* of **F** around C	$\iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S (\mathbf{F} \bullet \mathbf{n})\ dS$ The *flux* of **F** across S in direction **n**

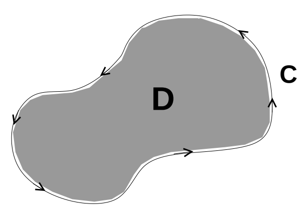
16. Integration Theorems:

Fundamental Theorem of Calculus: $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

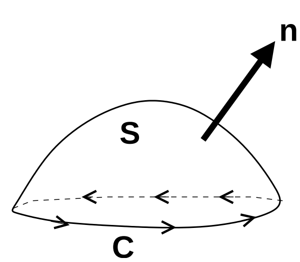
Fundamental Theorem of Calculus For Line Integrals: $\displaystyle\int_a^b \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$



Green's Theorem: $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \int_C P(x,y)\,dx + Q(x,y)\,dy$



Stokes' Theorem: $\displaystyle\iint_S \operatorname{curl}\mathbf{F} \bullet d\mathbf{S} = \int_C \mathbf{F} \bullet d\mathbf{r}$



Divergence Theorem: $\displaystyle\iiint_E \operatorname{div}\mathbf{F}\;\, dV = \iint_S \mathbf{F} \bullet d\mathbf{S}$