MA261-YOLCU PRACTICE PROBLEMS FOR TEST 2 SPRING 2013

(1) Consider $f(x,y) = x^4 + y^4 - 4xy + 1$. Show that f has a local minimum at $(1,1)$ and $(-1,-1)$ and that $(0,0)$ is a saddle point of f.

$f_x = 4x^3 - 4y$ $\quad$ $f_y = 4y^3 - 4x$

$f_x = 0 \Downarrow y = x^3$ $\quad$ & $\quad$ $f_y = 0 \Downarrow x = y^3$

$y = y^9$

$y(y^8 - 1) = 0$

$y(y^2-1)(y^2+1)(y^4+1) = 0$

$y = 0 \quad y = 1, \quad y = -1$

$x = y^3$

$x = 0 \quad x = 1 \quad x = -1$

$(0,0), (1,1), (-1,-1)$ are C.Ps

$f_{xx} = 12x^2 \quad f_{yy} = 12y^2 \quad f_{xy} = -4$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 144x^2y^2 - (-4)^2 = 144x^2y^2 - 16$

- $D(0,0) = -16 \Rightarrow (0,0)$ is a saddle point
- $D(1,1) = 144 - 16 = 128 > 0$

  $f_{xx}(1,1) = 12 > 0$

  $\Rightarrow (1,1)$ is a local minimum.
- $D(-1,-1) = 144 - 16 = 128 > 0$

  $f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$

  $\Rightarrow (-1,-1)$ is a local minimum.

(2) Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $\underbrace{x^2 + y^2}_{g(x,y) = x^2+y^2} = 1$. Answer: $f(0,\pm 1) = 2$ is the maximum, $f(\pm 1, 0) = 1$ is the minimum value.

Solve $\nabla f = \lambda \nabla g$, $x^2+y^2=1$ for x, y & λ.

$f_x = \lambda g_x$, $f_y = \lambda g_y$, $x^2+y^2=1$ $\Leftarrow$ $2x = \lambda(2x) \dots (1)$, $4y = \lambda(2y) \dots (2)$, $x^2+y^2 = 1 \dots (3)$

From (1) $\quad x(\lambda - 1) = 0 \Rightarrow x = 0$ or $\lambda = 1$

Case 1. $x = 0$. Then $x^2+y^2 = 1 \Leftrightarrow y^2 = 1 \Leftrightarrow y = \pm 1 \Rightarrow (0, \pm 1)$

Case 2: $\lambda = 1$: By (2) $\quad 4y = 2y \Rightarrow y = 0 \overset{(3)}{\Rightarrow} x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow (\pm 1, 0)$

$f(0,\pm 1) = 2, \quad f(\pm 1, 0) = 1$

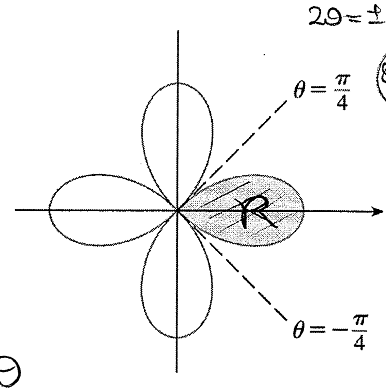
$\uparrow$ Max $\quad \uparrow$ min

(3) Find the area of **one loop** of the rose $r = \cos(2\theta)$ sketched below. Answer: $\dfrac{\pi}{8}$

$\cos(2\theta) = 0$

$2\theta = \pm\frac{\pi}{2}$

$\theta = \pm\frac{\pi}{4}$



$$\text{Area of } R = \iint_R dA = \int_{-\pi/4}^{\pi/4}\int_0^{\cos(2\theta)} r\,dr\,d\theta$$

$$= \int_{-\pi/4}^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\cos(2\theta)} d\theta$$

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \underbrace{\cos^2(2\theta)}_{\text{even in }\theta}\,d\theta = \int_0^{\pi/4}\cos^2(2\theta)\,d\theta$$

$\cos(2x) = 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\cos(4\theta)+1\right)d\theta = \frac{1}{2}\left[\frac{1}{4}\sin(4\theta)+\theta\right]_{\theta=0}^{\theta=\frac{\pi}{4}} = \frac{\pi}{8}.$$

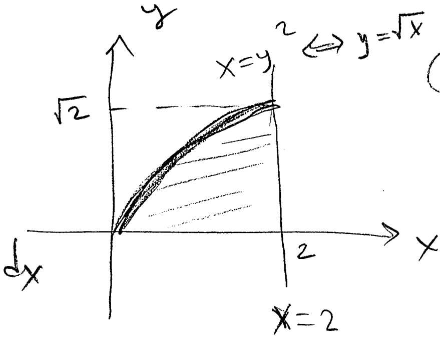
(4) Find the value of the integral $I = \displaystyle\int_0^{\sqrt{2}}\int_{y^2}^{2} y\,e^{x^2}\,dx\,dy$ by interchanging the order of integration.

Answer:

$$I = \int_0^2\int_0^{\sqrt{x}} y\,e^{x^2}\,dy\,dx = \frac{1}{4}(e^4 - 1).$$

See also similar problems on page: 996 (15.3(#49 – 54)).

$x = y^2 \Leftrightarrow y = \sqrt{x}$ b/c $y > 0$



$$I = \int_0^2\int_0^{\sqrt{x}} y\,e^{x^2}\,dy\,dx = \int_0^2 e^{x^2}\left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^2 e^{x^2}\cdot\frac{x}{2}\,dx = \frac{1}{4}\int_0^2 2x\,e^{x^2}\,dx = \frac{1}{4}\left[e^{x^2}\right]_{x=0}^{x=2}$$

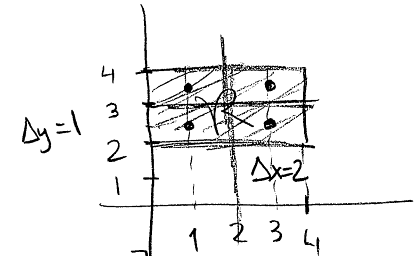
$$= \frac{1}{4}\left[e^4 - 1\right].$$

(5) Use the midpoint rule with $m = n = 2$ to approximate

$$\iint_R (x^2-1)y\,dA$$

where R is the region $\{(x,y) \ : \ 0 \le x \le 4,\ 2 \le y \le 4\}$.

Asnwer: 96



$\Delta A = \Delta x\,\Delta y = 2\cdot 1 = 2$

$f(x,y) = (x^2-1)y$

$$\iint_R (x^2-1)\,dA \approx \left[f(1,2.5) + f(1,3.5) + f(3,2.5) + f(3,3.5)\right]\Delta A$$

$$= \left[0 + 0 + 8\cdot(2.5) + 8\cdot 3.5\right]\cdot 2$$

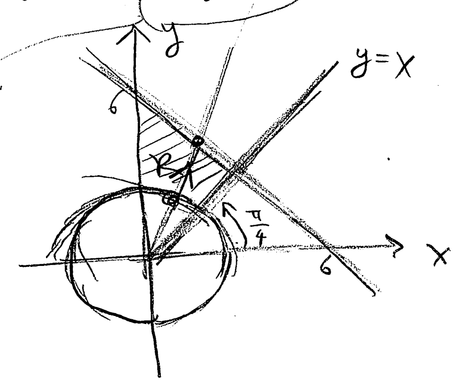
$$= \left[20 + 28\right]\cdot 2 = 48\cdot 2 = 96$$

(6) Let R be the region in the first quadrant bounded by $x = 0$, $x - y = 0$, $x^2 + y^2 = 9$ and $x + y = 6$. Evaluate

$r^2 = 9 \Rightarrow r = 3$

$$\iint_R \frac{x+y}{x^2+y^2}\,dA.$$

Answer: $\dfrac{3}{2}\pi - 3$



$x+y=6 \iff r\cos\theta + r\sin\theta = 6$

$\iff r = \dfrac{6}{\cos\theta + \sin\theta}$

$\dfrac{\pi}{4} \le \theta \le \dfrac{\pi}{2}$ $\qquad 3 \le r \le \dfrac{6}{\cos\theta+\sin\theta}$

$$\iint_R \frac{x+y}{x^2+y^2}\,dA = \int_{\pi/4}^{\pi/2}\int_3^{\frac{6}{\cos\theta+\sin\theta}} \frac{r\cos\theta + r\sin\theta}{r^2}\,(r\,dr\,d\theta)$$

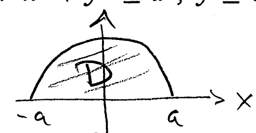
$$= \int_{\pi/4}^{\pi/2} (\sin\theta + \cos\theta)\left(\frac{6}{\sin\theta+\cos\theta} - 3\right) d\theta$$

$$= \int_{\pi/4}^{\pi/2} (6 - 3\sin\theta - 3\cos\theta)\,d\theta = \frac{3}{2}\pi - 3.$$

Exercise

(7) Find the center of mass $(\bar{x}, \bar{y})$ of the semicircular lamina described by $\{(x,y) : x^2+y^2 \le a^2,\ y \ge 0\}$ if its density at the point (x,y) is $\rho(x,y) = \sqrt{x^2+y^2}$.

Answer: $\bar{x} = 0$, $\bar{y} = \dfrac{3a}{2\pi}$



$$m = \text{mass} = \iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^{a} r\,(r\,dr\,d\theta) = \left(\int_0^{\pi} d\theta\right)\left(\int_0^{a} r^2\,dr\right) = \pi\frac{a^3}{3}.$$

$$\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA = \frac{3}{\pi a^3}\int_0^{\pi}\int_0^{a}(r\cos\theta)\,r\cdot(r\,dr\,d\theta) = \frac{3}{\pi a^3}\underbrace{\left(\int_0^{\pi}\cos\theta\,d\theta\right)}_{=0}\left(\int_0^{a} r^3\,dr\right) = 0$$

$$\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA = \frac{3}{\pi a^3}\int_0^{\pi}\int_0^{a}(r\sin\theta)\cdot r\cdot(r\,dr\,d\theta) = \frac{3}{\pi a^3}\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{a} r^3\,dr\right)$$

$$\Rightarrow \bar{y} = \frac{3}{\pi a^3}\left(\Big[-\cos\theta\Big]_{\theta=0}^{\theta=\pi}\right)\left(\Big[\frac{r^4}{4}\Big]_{r=0}^{r=a}\right) = \frac{3a}{2\pi} \qquad \Rightarrow \quad \bar{x} = 0,\ \ \bar{y} = \frac{3a}{2\pi}$$

(8) Find the area of the region described by the intersection of two disks bounded by $x^2+y^2 = x$ and $x^2+y^2 = y$.

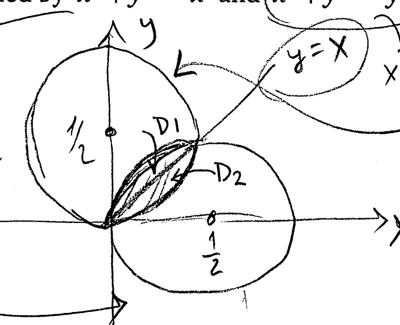
Answer: $\dfrac{\pi}{8} - \dfrac{1}{4}$

$$\left.\begin{array}{l} x^2+y^2 = x \\ x^2+y^2 = y \end{array}\right\} \Rightarrow y = x$$

$$\underbrace{x^2 - x + \tfrac{1}{4}}\, + y^2 = \tfrac{1}{4}$$

$$\left(x-\tfrac{1}{2}\right)^2 + y^2 = \left(\tfrac{1}{2}\right)^2$$

$$x^2 + \left(y-\tfrac{1}{2}\right)^2 = \left(\tfrac{1}{2}\right)^2$$



Area of $D_1 \& D_2 = 2 \times$ Area of D_2
by Symmetry

$r^2 = r\sin\theta$

$r = \sin\theta$

$$= 2\iint_{D_2} dA = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta$$

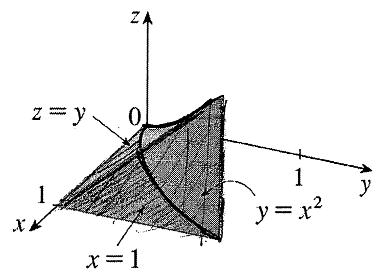
$$= \cancel{2}\int_0^{\pi/4}\left[\frac{r^2}{\cancel{2}}\right]_{r=0}^{r=\sin\theta} d\theta = \int_0^{\pi/4}\sin^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left(1-\cos(2\theta)\right)d\theta$$

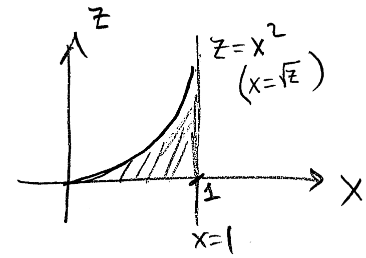
$$= \frac{1}{2}\underbrace{\left(\left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{\theta=\frac{\pi}{4}}\right)}_{=\left(\frac{\pi}{4}-\frac{1}{2}\right)} = \frac{\pi}{8} - \frac{1}{4}$$

(9) Find a,b,c,d,e,f,g,h so that

$$\int_0^1\int_0^y\int_{\sqrt{y}}^1 F(x,y,z)\,dx\,dz\,dy=\int_0^1\int_a^b\int_c^d F(x,y,z)\,dy\,dx\,dz=\int_0^1\int_e^f\int_g^h F(x,y,z)\,dz\,dy\,dx$$

Answer: $a=\sqrt{z},\ b=1,\ c=z,\ d=x^2,\ e=0, f=x^2,\ g=0,\ h=y$. Furthermore, the following six integrals are the same:

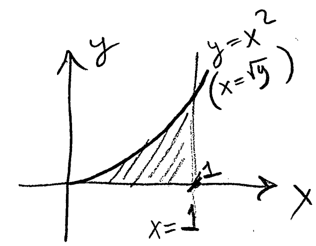




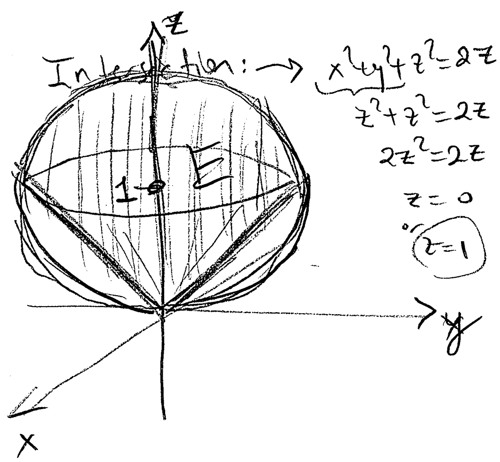
$$\int_0^1\int_0^y\int_{\sqrt{y}}^1 F(x,y,z)\,dx\,dz\,dy=\int_0^1\int_z^1\int_{\sqrt{y}}^1 F(x,y,z)\,dx\,dy\,dz$$

$$\int_0^1\int_{\sqrt{z}}^1\int_z^{x^2} F(x,y,z)\,dy\,dx\,dz=\int_0^1\int_0^{x^2}\int_z^{x^2} F(x,y,z)\,dy\,dz\,dx$$

$$\int_0^1\int_0^{x^2}\int_0^y F(x,y,z)\,dz\,dy\,dx=\int_0^1\int_{\sqrt{y}}^1\int_0^y F(x,y,z)\,dz\,dx\,dy$$



(10) Find the volume the solid E that lies above the cone $z=\sqrt{x^2+y^2}$ below the sphere $x^2+y^2+z^2=2z$.
Answer: π



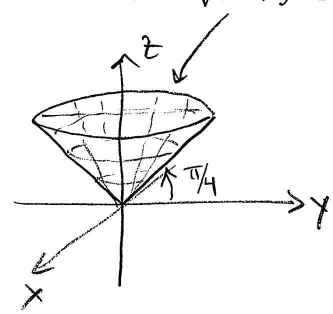
Intersection: $\rightarrow$ $\underbrace{x^2+y^2}+z^2=2z$

$z^2+z^2=2z$

$2z^2=2z$

$z=0$

or $z=1$

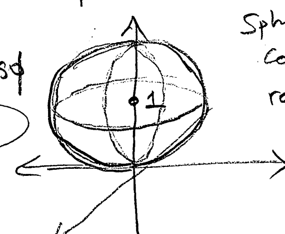


$x^2+y^2+(z-1)^2=1^2$

Sphere
Center $(0,0,1)$
radius = 1

$\rho^2=2\rho\cos\phi$

$\rho=2\cos\phi$



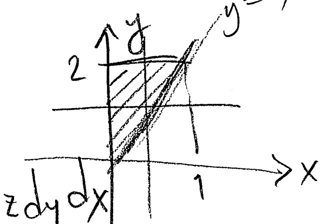
$$\text{Volume of } E=\iiint_E dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

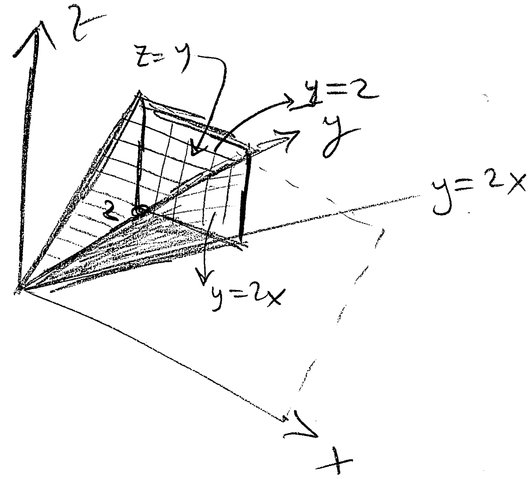
$$=(2\pi)\cdot\int_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=2\cos\phi}\sin\phi\,d\phi\;=\;2\pi\cdot\frac{8}{3}\int_0^{\pi/4}\cos^3\phi\,\sin\phi\,d\phi$$

$$=\frac{\cancel{16}^{\,4}\pi}{3}\left[-\frac{\cos^4\phi}{\cancel{4}}\right]_{\phi=0}^{\phi=\frac{\pi}{4}}\;=\;\frac{4\pi}{3}\left[-\left(\frac{1}{\sqrt{2}}\right)^4-(-1)\right]=\frac{4\pi}{3}\left[1-\frac{1}{4}\right]=\pi.$$

(11) Let T be the solid region in the first octant that is bounded by the planes $y = 2$, $x = 0$, $y = 2x$, $z = 0$, and $z = 2y$. What is the value of the tripple integral $\iiint_T x\,dV$?

Answer: 1



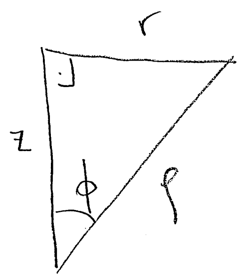


$$\iiint_T x\,dV = \int_0^1 \int_{2x}^2 \int_0^{2y} x\,dz\,dy\,dx$$

or

$$= \int_0^2 \int_0^{\frac{y}{2}} \int_0^{2y} x\,dz\,dx\,dy = 1$$

(12) Let c be a constant such that $0 < c < \pi/2$ or $\pi/2 < c < \pi$. Show that the equation of the surface $\phi = c$ converted to rectangular coordinates becomes $z = \cot(c)\sqrt{x^2 + y^2}$.



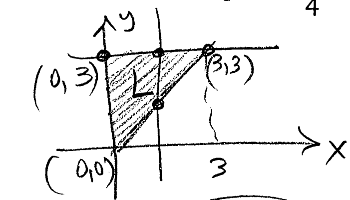
$$\tan\phi = \frac{r}{z} = \frac{\sqrt{x^2+y^2}}{z}$$

$$z = \frac{\sqrt{x^2+y^2}}{\tan(c)} = \cot(c)\sqrt{x^2+y^2},$$

(13) A lamina L occupies the triangular region in the xy–plane with vertices $(0,0)$, $(0,3)$ and $(3,3)$. If the mass density at (x,y) is $\rho(x,y) = x+2y$, then show that the y–coordinate of the center of mass of L is equal to $\frac{9}{4}$.

$$m = \iint_L (x+2y)\,dA = \int_0^3 \int_x^3 (x+2y)\,dy\,dx = \int_0^3 \Big[xy + y^2\Big]_{y=x}^{y=3} dx = \frac{45}{2}$$

Exercise



$x \le y \le 3$
$0 \le x \le 3$

$$\bar{y} = \frac{1}{m}\iint_L y\,\rho(x,y)\,dA = \frac{1}{\left(\frac{45}{2}\right)} \int_0^3 \int_x^3 (xy + 2y^2)\,dy\,dx$$

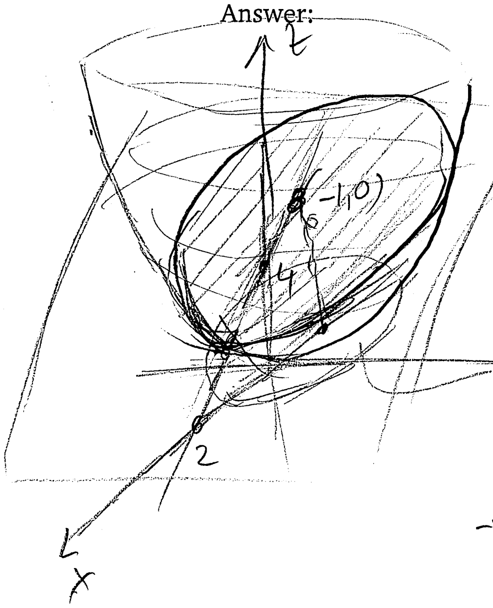
$$= \frac{2}{45}\int_0^3 \Big[x\frac{y^2}{2} + \frac{2}{3}y^3\Big]_{y=x}^{y=3} dx = \frac{2}{45}\int_0^3 \left(\frac{9}{2}x + 18 - \frac{7}{6}x^3\right) dx = \frac{9}{4}.$$

Exercise

lower bound   upper bound

(14) Let E be the solid region enclosed by the paraboloid $z = x^2 + y^2$ and the plane $2x + z = 4$. Find the triple integral in cylindrical coordinates and rectangular coordinates that give the volume $V(E)$ of solid E.

Answer:



$$V(E) = \int_0^{2\pi} \int_0^{-\cos\theta + \sqrt{4+\cos^2\theta}} \int_{r^2}^{4-2r\cos\theta} r\,dz\,dr\,d\theta$$

$$V(E) = \int_{-1-\sqrt{5}}^{-1+\sqrt{5}} \int_{-\sqrt{4-2x-x^2}}^{\sqrt{4-2x-x^2}} \int_{x^2+y^2}^{4-2x} dz\,dy\,dx$$

$x^2+y^2 = z = 4-2x$

$x^2+y^2 = 4-2x$

$(x+1)^2 + y^2 = 5$

$r^2 = 4 - 2r\cos\theta$

$r^2 + 2\cos\theta\, r - 4 = 0$

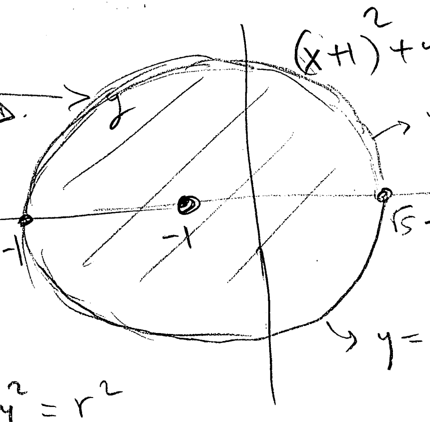
$r > 0$

$r = \dfrac{-2\cos\theta \pm \sqrt{4\cos^2\theta + 16}}{2}$

$r = -\cos\theta + \sqrt{\cos^2\theta + 4}$

$(x+1)^2 + y^2 = (\sqrt{5})^2$

$y = \sqrt{4-2x-x^2}$



$y = -\sqrt{5-(x+1)^2} = -\sqrt{4-2x-x^2}$

Lower bound $z = x^2+y^2 = r^2$

upper bound $z = 4-2x = 4-2r\cos\theta$

$r^2 \le z \le 4 - 2r\cos\theta$

$0 \le \theta \le 2\pi$

$0 \le r \le -\cos\theta + \sqrt{4+\cos^2\theta}$

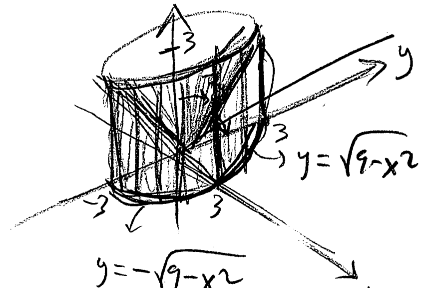
(15) Convert

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{x^2+y^2}} x\,y\,z\,dz\,dy\,dx$$

to spherical coordinates.

Answer:

$$\int_{-\pi/2}^{\pi/2} \int_{\pi/4}^{\pi/2} \int_0^{3/\sin\phi} \rho^5 \cos\phi\,\sin^3\phi\,\cos\theta\,\sin\theta\,d\rho\,d\phi\,d\theta$$



$$xyz = \rho\sin\phi\cos\theta\;\rho\sin\phi\sin\theta\;\rho\cos\phi = \rho^3\sin^2\phi\cos\phi\,\sin\theta\cos\theta$$

$$(xyz)dV = (\rho^3\sin^2\phi\,\cos\phi\,\sin\theta\,\cos\theta)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$$

$$xyz\,dV = \rho^5\sin^3\phi\,\cos\phi\,\sin\theta\,\cos\theta\,d\rho\,d\phi\,d\theta$$

$\boxed{x^2+y^2=9}$

$z=\sqrt{x^2+y^2}=\sqrt{9}=3$

ρ is bounded by cylinder $\underbrace{x^2+y^2}=9$

$\rho^2\sin^2\phi=9$

$0\le\rho\le\frac{3}{\sin\phi}$

$\frac{\pi}{4}\le\phi\le\frac{\pi}{2}$

$-\pi/2\le\theta\le\pi/2$

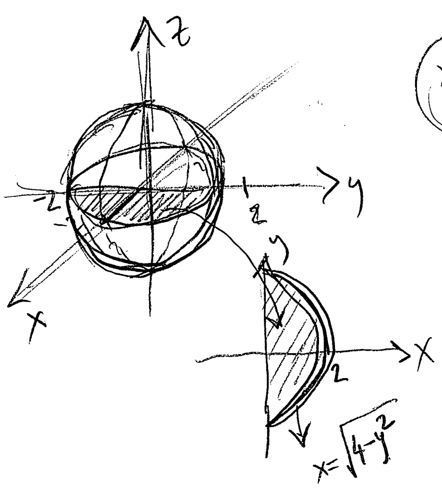
$\sin^2 x=\frac{1-\cos(2x)}{2}$

(16) What is the value of $I=\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$?

Answer: $\frac{64\pi}{9}$.

$x^2+y^2+z^2=4$

$x\ge0$



$$I=\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2}(\rho^2\sin^2\phi\sin^2\theta)\cdot\rho\cdot(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

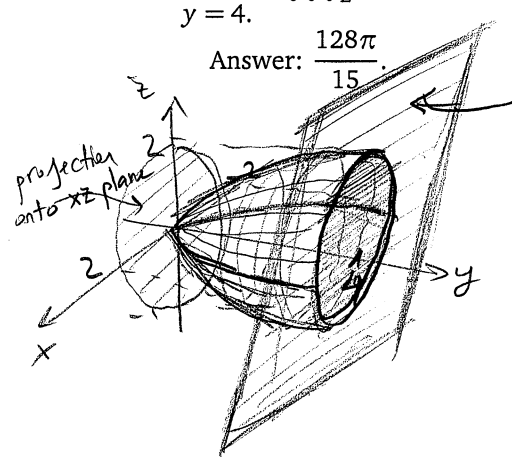
y^2, $\sqrt{x^2+y^2+z^2}$

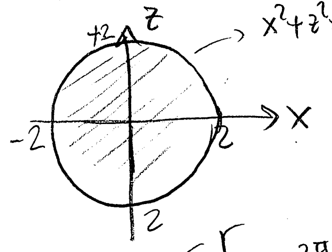
$$=\left(\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^{2}\rho^5\,d\rho\right)$$

$$=\frac{\pi}{2}\cdot\frac{4}{3}\cdot\frac{32}{3}=\frac{64\pi}{9}.$$

(17) Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y=x^2+z^2$ and the plane $y=4$.

Answer: $\frac{128\pi}{15}$.





$x^2+z^2=4=2^2$

$x=r\cos\theta$

$z=r\sin\theta$

$dx\,dz=r\,dr\,d\theta$

$r^2=x^2+z^2$

$$\iiint_E \sqrt{x^2+z^2}\,dV=\int_0^{2\pi}\int_0^{2}\int_{r^2}^{4} r\cdot(r\,dy\,dr\,d\theta)$$

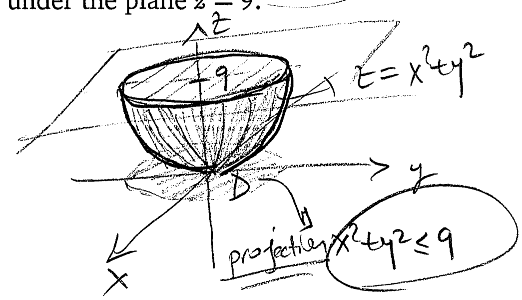
$$=(2\pi)\int_0^2 r^2(4-r^2)\,dr = 2\pi\left[\frac{4}{3}r^3-\frac{r^5}{5}\right]_{r=0}^{r=2} = 2\pi\cdot\underbrace{\left[\frac{32}{3}-\frac{32}{5}\right]}_{=\frac{64}{15}}$$

$$=\frac{128\pi}{15}$$

(18) Find the surface area of the part of the paraboloid $z=x^2+y^2$ that lies under the plane $z=9$.

Answer: $\frac{\pi}{6}(37\sqrt{37}-1)$.

$f(x,y)=x^2+y^2$



$$\text{Surface area in question} = \iint_D \sqrt{f_x^2+f_y^2+1}\,dA$$

D, $x^2+y^2\le 9$

$$=\iint_{x^2+y^2\le 9} \sqrt{(2x)^2+(2y)^2+1}\,dA$$

$$=\int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\;r\,dr\,d\theta$$

$$=\cancel{2}\pi\int_0^3 \frac{1}{\cancel{8}_{4}}(8r)\sqrt{4r^2+1}\,dr = \frac{\pi}{4}\left[\frac{(4r^2+1)^{3/2}}{3/2}\right]_{r=0}^{r=3}$$

$$=\frac{\pi}{6}\left[37\sqrt{37}-1\right],$$

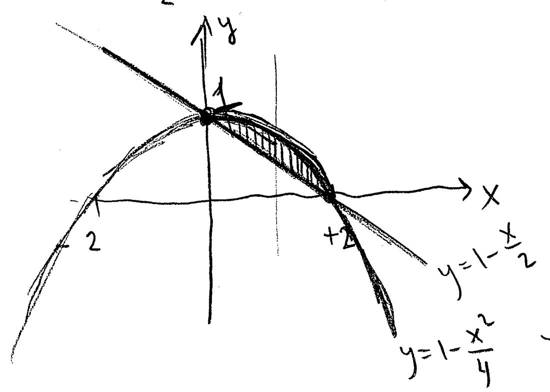
(19) Find K, L, M, and N so that

$$\int_0^2 \int_{1-(x/2)}^{1-(x^2/4)} f(x,y)\,dy\,dx = \int_K^L \int_M^N f(x,y)\,dx\,dy.$$

Answer: $K = 0$, $L = 1$, $M = 2 - 2y$, $N = 2\sqrt{1-y}$.

$1 - \frac{x}{2} \leq y \leq 1 - \frac{x^2}{4}$, $0 \leq x \leq 2$

$x > 0$



line $\leq x \leq$ parabola

$2 - 2y = 2(1-y) \leq x \leq \sqrt{4(1-y)} = 2\sqrt{1-y}$

$0 \leq y \leq 1$

Integral in question $= \int_0^1 \int_{2-2y}^{2\sqrt{1-y}} f(x,y)\,dx\,dy$

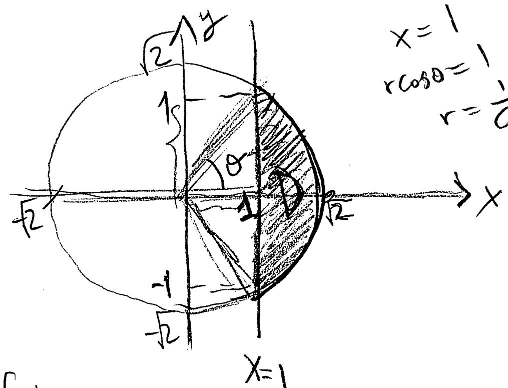
(20) Write the double integral in polar coordinates representing the area of the planar region bounded on the right by $x^2 + y^2 = 2$ and bounded on the left by $x = 1$.

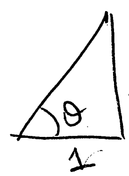
Answer: $\displaystyle\int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r\,dr\,d\theta = \frac{\pi}{2} - 1.$

$\left.\begin{array}{l} x^2 + y^2 = 2 \\ x = 1 \end{array}\right\} \Rightarrow 1^2 + y^2 = 2$

$y = \pm 1$

$x = 1$
$r\cos\theta = 1$
$r = \frac{1}{\cos\theta} = \sec\theta$





$\Rightarrow \theta = \frac{\pi}{4}$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$\sec\theta \leq r \leq \sqrt{2}$

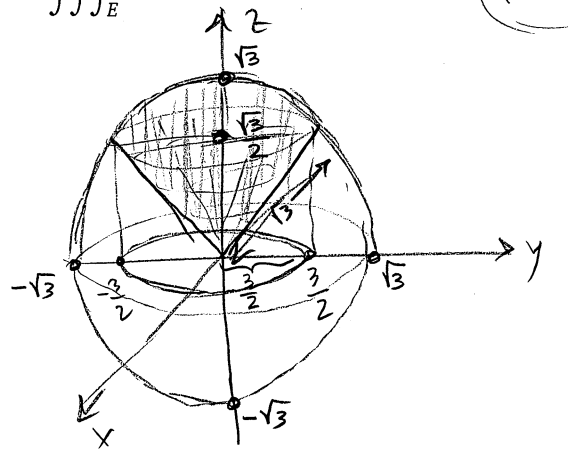
Area $= \iint_D dA$

$$\text{Area} = \int_{-\pi/4}^{\pi/4} \int_{\sec\theta}^{\sqrt{2}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} \left[\frac{r^2}{2}\right]_{r=\sec\theta}^{r=\sqrt{2}} d\theta = \int_{-\pi/4}^{\pi/4} \left[1 - \frac{1}{2}\sec^2\theta\right] d\theta$$

$$= \left[\theta - \frac{1}{2}\tan\theta\right]_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \left[\frac{\pi}{4} - \frac{1}{2}\underbrace{\tan\left(\frac{\pi}{4}\right)}_{=1}\right] - \left[-\frac{\pi}{4} - \frac{1}{2}\underbrace{\tan\left(-\frac{\pi}{4}\right)}_{=-1}\right]$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\right] - \left[-\frac{\pi}{4} + \frac{1}{2}\right] = \left(\frac{\pi}{4} + \frac{\pi}{4}\right) - \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{\pi}{2} - 1$$

(21) (2006 Fall# 3) Let E be the solid region bounded below by $z = \sqrt{\dfrac{x^2+y^2}{3}}$ (cone) and above by $x^2+y^2+z^2 = 3$ (sphere). Write $\iiint_E z\, dV$ in spherical coordinates. & compute !



$x^2+y^2+z^2=3 \Leftrightarrow \rho^2=3 \Leftrightarrow \rho=\sqrt{3}$

$0 \le \rho \le \sqrt{3}$

$3z^2 = x^2+y^2$ &

$3z^2+z^2=3$ $\quad z>0$

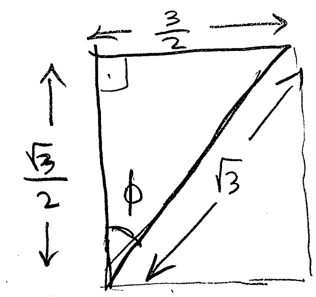
$z^2=\frac{3}{4}$

$\boxed{z=\frac{\sqrt{3}}{2}}$

$\left(\frac{3}{2}\right)^2 = 3\cdot\left(\frac{3}{4}\right) = x^2+y^2$

$x^2+y^2 = \left(\frac{3}{2}\right)^2$

$r=\frac{3}{2}$



$\tan\phi = \dfrac{\frac{3}{2}}{\frac{\sqrt{3}}{2}} = \sqrt{3}$

$\Rightarrow \phi = \frac{\pi}{3}$

$0 \le \theta \le 2\pi$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$$\iiint_E z\,dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\sqrt{3}} (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3}\int_0^{\sqrt{3}} \rho^3 \sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\left(\int_0^{\pi/3}\sin\phi\cos\phi\,d\phi\right)\left(\int_0^{\sqrt{3}}\rho^3\,d\rho\right)$$

$$= \cancel{2}\pi \cdot \left[\frac{\sin^2\phi}{\cancel{2}}\right]_{\phi=0}^{\phi=\frac{\pi}{3}} \cdot \left[\frac{\rho^4}{4}\right]_{\rho=0}^{\rho=\sqrt{3}}$$

$$= \pi\cdot\left(\frac{\sqrt{3}}{2}\right)^2 \cdot \frac{(\sqrt{3})^4}{4} = \pi\cdot\frac{3}{4}\cdot\frac{9}{4} = \frac{27\pi}{16}.$$