Fourier Series

Some Preliminary Ideas:

- *Odd/Even Functions:*
  - Sine is odd, which means $\sin(-x) = -\sin x$
  - Cosine is even, which means $\cos(-x) = \cos x$
- *Special values of siine an cosine at $n\pi$*
  - When dealing with series, n is always a positive integer. Remember at every π, sine has a value of zero, which means

$$\sin n\pi = 0$$

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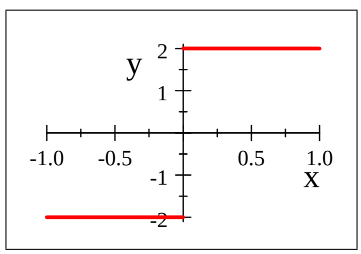
  - Cosine, on the otherhand, alternates between 1 and -1. So at odd values of n, $\cos n\pi = -1$ and at even values of n, $\cos n\pi = 1$, which means

$$\cos n\pi = (-1)^n$$

1 What is a Fourier series?

The Fourier series are useful for describing periodic phenomena. The advantage that the Fourier series has over Taylor series is that the function itself does not need to be continuous. Take for example a square wave defined by one period as

$$f(x) = \begin{cases} -2 & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$$



Could this easily be approximated using a polynomial, like we did using Taylor series? Probably not very well. Since this a periodic function (only one period shown), it might be more useful to use periodic functions such as sine and cosine. This is exactly what the Fourier series does. The Fourier series is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) \text{ for a function defined on the interval } (-p, p)$$

where

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx$$

Looks like fun, right? What we are going to consider are two special cases. The first is when $f(x)$ is a constant function, and the second is when $f(x)$ is a linear function.

2 Case 1 $f(x)$ is a constant.

Consider $f(x) = \{ \; k \quad a < x < b$, where a and b are numbers in $[-p, p], a \leq b$, and k is any real number.

NOTE: The reason I am using a and b for the bounds is that the function might be broken into individual pieces within a piecewise defined function, and you would take the integrals individually. You will see this in the examples.

2.1 Finding a_0

Finding a_0 : $a_0 = \frac{1}{p}\int_a^b f(x)dx$

Substituting in $f(x) = k$ you get

$$\frac{1}{p}\int_a^b kdx = \frac{1}{p}(kx)_{x=a}^{x=b} = \frac{k}{p}(b-a)$$

2.2 Finding a_n

Finding a_n : $a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}dx$

For now I am going to ignore the bounds and concentrate on the integral itself:

$$\frac{1}{p}\int f(x)\cos\frac{n\pi x}{p}dx$$

Plugging in $f(x) = k$ we get

$$\frac{1}{p}\int k\cos\frac{n\pi x}{p}dx$$

Then we can do u-substitution:

$$u = \frac{n\pi x}{p} \quad \rightarrow \quad du = \frac{n\pi}{p}dx$$
$$dx = \frac{p}{n\pi}du$$

Substituting in

$$\frac{1}{p}\int k\cos u\left(\frac{p}{n\pi}du\right) = \frac{1}{p}\left(\frac{kp}{n\pi}\right)(\sin u) = \frac{k}{n\pi}\sin\frac{n\pi x}{p}$$

Now plug in the bounds:

$$\frac{k}{n\pi}\sin\frac{n\pi x}{p}|_{x=z}^{x=b} = \frac{k}{n\pi}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

So all that you need to do now is plug in the values for a, b, k, and p.

2.3 Finding b_n

Finding b_n : $b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}dx$

Similar to above, I am going to ignore the bounds for now and plug them back in at the end.

$$b_n = \frac{1}{p}\int f(x)\sin\frac{n\pi x}{p}dx$$

Again, plugging in $f(x) = x$ you get

$$\frac{1}{p}\int k\sin\frac{n\pi x}{p}dx$$

Then we can do u-substitution: $u = \frac{n\pi x}{p} \quad \rightarrow \quad du = \frac{n\pi}{p}dx$

$$dx = \frac{p}{n\pi}du$$

Substituting in

$$\frac{1}{p}\int k\sin u\left(\frac{p}{n\pi}du\right) = \frac{k}{n\pi}\int \sin u du = \frac{k}{n\pi}(-\cos u) = \frac{k}{n\pi}\left(-\cos\frac{bn\pi}{p} - \left(-\cos\frac{an\pi}{p}\right)\right)$$

$$= \frac{k}{n\pi}\left(\cos\frac{an\pi}{p} - \cos\frac{bn\pi}{p}\right)$$

Then finish by substituting back in for $a, b, k,$ and p.

2.4 Summary:

For any constant function defined by $f(x) = k$ on and interval $(a,b) \subseteq (-p,p)$, the coefficients of the Fourier series can be determined by

$$a_0 = \frac{k}{p}(b-a)$$

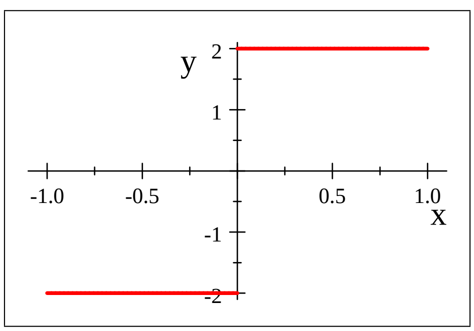
$$a_n = \frac{k}{n\pi}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

$$b_n = \frac{k}{n\pi}\left(\cos\frac{an\pi}{p} - \cos\frac{bn\pi}{p}\right)$$

2.5 Examples Where $f(x)$ is a constant

Now let's look at some examples, starting with the one listed at the beginning.

Example 1 $f(x) = \begin{cases} -2 & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$



First note that $p = 1$ (the entire length of the period is -1 to 1, and p is always half of that)

The function is also broken up onto 2 parts: from -1 to 0, $f(x) = -2$, and from 0 to 1, $f(x) = 2$. This means that each section of this function will be its own separate integral. So for a_0, we will have:

$$a_0 = \frac{1}{1}\int_{-1}^{0} -2dx + \frac{1}{1}\int_{0}^{1} 2dx$$

But we can take advantage of the formulas given for this, we just need to do it for each interval then add them together:

- $a_0 = \frac{k}{p}(b-a)$

Interval 1: $(-1,0)$	Interval 2: $(0,1)$
$a=-1, b=0, k=-2, p=1$	$a=0, b=1, k=2, p=1$
$\frac{-2}{1}(0-(-1)) = -2$	$\frac{2}{1}(1-0) = 2$

So to find a_0 add the two values together:
$a_0 = -2+2 = 0$

It is similar for finding a_n and b_n. Calculate each separately then add them together.

- $a_n = \frac{k}{n\pi}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$

Interval 1: $(-1,0)$	Interval 2: $(0,1)$
$a=-1, b=0, k=-2, p=1$	$a=0, b=1, k=2, p=1$
$\frac{-2}{n\pi}\left(\sin\frac{0n\pi}{1} - \sin\frac{-1n\pi}{1}\right)$	$\frac{2}{n\pi}\left(\sin\frac{1n\pi}{1} - \sin\frac{0n\pi}{1}\right)$
$= \frac{-2}{n\pi}(\sin 0 - \sin(-n\pi))$	$= \frac{2}{n\pi}(\sin n\pi - \sin 0)$
$= \frac{-2}{n\pi}(-\sin(-n\pi))$	$= \frac{2}{n\pi}\sin n\pi$

So that means

$a_n = \frac{2}{n\pi}\sin(-n\pi) + \frac{2}{n\pi}\sin(n\pi)$

However...note that since n is always in integer, there will always be a whole value of π inside each value of sine, and since $\sin\pi = 0$, then any $\sin(n\pi) = 0$, therefore

$a_n = \frac{2}{n\pi}(0) + \frac{2}{n\pi}(0) = 0$

- $b_n = \frac{k}{n\pi}\left(\cos\frac{an\pi}{p} - \cos\frac{bn\pi}{p}\right)$

Interval 1: $(-1,0)$	Interval 2: $(0,1)$
$a=-1, b=0, k=-2, p=1$	$a=0, b=1, k=2, p=1$
$\frac{-2}{n\pi}\left(\cos\frac{-1n\pi}{1} - \cos\frac{0n\pi}{1}\right)$	$\frac{2}{n\pi}\left(\cos\frac{0n\pi}{1} - \cos\frac{1n\pi}{1}\right)$
$= \frac{-2}{n\pi}(\cos(-n\pi) - \cos(0))$	$= \frac{2}{n\pi}(\cos(0) - \cos n\pi)$
$= \frac{-2}{n\pi}(\cos(-n\pi) - 1)$	$= \frac{2}{n\pi}(1 - \cos n\pi)$

Which means that

$b_n = \frac{-2}{n\pi}(\cos(-n\pi) - 1) + \frac{2}{n\pi}(1 - \cos n\pi)$

Now let's do some algebra.

(1) Remember that cosine is even, so $\cos(-\theta) = \cos\theta$, so the negative in the first expression disappears.

$$b_n = \frac{-2}{n\pi}(\cos n\pi - 1) + \frac{2}{n\pi}(1 - \cos n\pi)$$

(2) Distribute the negative in the first expression to reverse the insides:

$$b_n = \frac{2}{n\pi}(1 - \cos n\pi) + \frac{2}{n\pi}(1 - \cos n\pi)$$

(3) Factor out $\dfrac{2}{n\pi}$

$$b_n = \frac{2}{n\pi}(1 - \cos n\pi + 1 - \cos n\pi) = \frac{2}{n\pi}(2 - 2\cos n\pi)$$

(4) Factor out a 2

$$b_n = \frac{4}{n\pi}(1 - \cos n\pi)$$

(5) **Tricky part**: Remember that cosine alternates between 1 and -1 at every other π, so $\cos n\pi = (-1)^2$

$$b_n = \frac{4}{n\pi}(1 - (-1)^n)$$

Which gives:

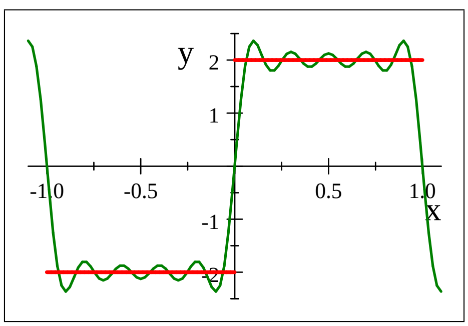
$a_0 = 0 \qquad a_n = 0 \qquad b_n = \dfrac{2}{n\pi}(1 - (-1)^n)$

FINAL STEP: Plug coefficients into the Fourier series:

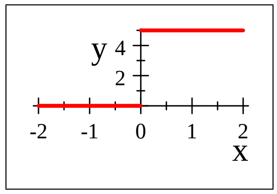
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\cos\frac{n\pi x}{1} + \left(\frac{4}{n\pi}(1 - (01)^n)\right)\sin\frac{n\pi x}{1}\right)$$

$$\boxed{= \sum_{n=1}^{\infty}\left(\left(\frac{4}{n\pi}(1 - (-1)^n)\right)\sin n\pi x\right)}$$



Example 2 $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 5 & 0 < x < 2 \end{cases}$



So this means that $p = 2$. Again, this is broken into 2 intervals:

$(-2, 0)\,, k = 0 \qquad (0, 2)\,, k = 5$

But since $k = 0$ on the interval $(-2, 0)\,,$ all of the integrals on that interval will be zero. (Since all terms are multiplied by k, zero times anything is zero)

So we only need to pay attention to the interval $(0, 2)\,,$ where $a = 0, b = 2, k = 5, p = 2$

- Finding $a_0 = \frac{k}{p}(b - a)$

$a_0 = \frac{5}{2}(2 - 0) = 5$

- Finding $a_n = \frac{k}{n\pi}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$

$a_n = \frac{5}{n\pi}\left(\sin\frac{2n\pi}{2} - \sin\frac{0n\pi}{2}\right) = \frac{5}{n\pi}(\sin 2n\pi - 0)$

But remember, $\sin n\pi = 0$

so $a_n = 0$

- Find $b_n = \frac{k}{n\pi}\left(\cos\frac{an\pi}{p} - \cos\frac{bn\pi}{p}\right)$

$b_n = \frac{5}{n\pi}\left(\cos 0 - \cos\frac{2n\pi}{2}\right) = \frac{5}{n\pi}(1 - \cos n\pi)$

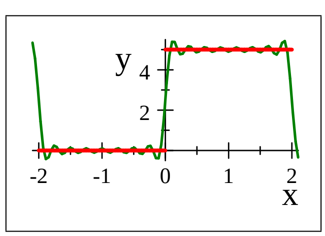
But remember, $\cos n\pi = (-1)^n$

so $b_n = \frac{5}{n\pi}(1 - (-1)^n)$

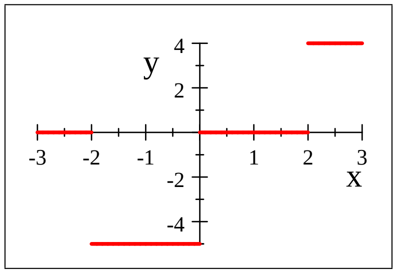
So plugging into the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)$ you get

$f(x) = \frac{5}{2} + \sum_{n=1}^{\infty}\left((0)\cos\frac{n\pi x}{2} + \left(\frac{5}{n\pi}(1 - (-1)^n)\right)\sin\frac{n\pi x}{2}\right)$

$$\boxed{f(x) = \frac{5}{2} + \sum_{n=1}^{\infty}\left(\frac{5}{n\pi}(1 - (-1)^n)\right)\sin\frac{n\pi x}{2}}$$



Example 3 $f(x) = \begin{cases} 0 & -3 < x < -2 \\ -5 & -2 < x < 0 \\ 0 & 0 < x < 2 \\ 4 & 2 < x < 3 \end{cases}$



Now the period has been broken into 4 intervals and $p = 3$ (half the length of the period):

Interval 1	Interval 2	Interval 3	Interval 4
$(-3,-2)\,, k=0$	$(-2,0)\,, k=-5$	$(0,2)\,, k=0$	$(2,3)\,, k=4$

We can ignore intervals 1 and 3, since $k = 0$ (so all of the integrals will be zero).

- $a_0 = \dfrac{k}{p}(b-a)$

Interval 2:
$a=-2, b=0, k=-5, p=3$

$$\frac{-5}{3}(0-(-2)) = -\frac{10}{3}$$

Interval 4:
$a=2, b=3, k=4, p=3$

$$\frac{4}{3}(3-2) = \frac{4}{3}$$

So $a_0 = -\dfrac{10}{3} + \dfrac{4}{3} = \boxed{-2}$

- $a_n = \dfrac{k}{n\pi}\left(\sin\dfrac{bn\pi}{p} - \sin\dfrac{an\pi}{p}\right)$

Interval 2:
$a=-2, b=0, k=-5, p=3$

$$\frac{-5}{n\pi}\left(\sin\frac{0n\pi}{3} - \sin\frac{-2n\pi}{3}\right)$$
$$\frac{-5}{n\pi}\left(\sin 0 - \sin\frac{-2n\pi}{3}\right)$$
$$\frac{-5}{n\pi}\left(0 - \left(-\sin\frac{2n\pi}{3}\right)\right)$$
$$\frac{-5}{n\pi}\sin\frac{2n\pi}{3}$$

Interval 4:
$a=2, b=3, k=4, p=3$

$$\frac{4}{n\pi}\left(\sin\frac{3n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$
$$\frac{4}{n\pi}\left(\sin n\pi - \sin\frac{2n\pi}{3}\right)$$
$$\frac{4}{n\pi}\left(-\sin\frac{2n\pi}{3}\right)$$
$$\frac{-4}{n\pi}\sin\frac{2n\pi}{3}$$

*Note in the last step, for interval 2, since sine is an odd function, $\sin(-\theta) = -\sin\theta$, so $\sin\dfrac{-2n\pi}{3} = -\sin\dfrac{2n\pi}{3}$, and for interval 4, remember that $\sin n\pi = 0$

$$a_n = \frac{-5}{n\pi}\sin\frac{2n\pi}{3} + \frac{-4}{n\pi}\sin\frac{2n\pi}{3}$$

$$a_n = \boxed{-\frac{9}{n\pi}\sin\frac{2n\pi}{3}}$$

- $b_n = \frac{k}{n\pi}\left(\cos\frac{an\pi}{p} - \cos\frac{bn\pi}{p}\right)$

Interval 2:
$a = -2, b = 0, k = -5, p = 3$

$\frac{-5}{n\pi}\left(\cos\frac{-2n\pi}{3} - \cos\frac{0n\pi}{3}\right)$

$\frac{-5}{n\pi}\left(\cos\frac{2n\pi}{3} - 1\right)$

Interval 4:
$a = 2, b = 3, k = 4, p = 3$

$\frac{4}{n\pi}\left(\cos\frac{2n\pi}{3} - \cos\frac{3n\pi}{3}\right)$

$\frac{4}{n\pi}\left(\cos\frac{2n\pi}{3} - (-1)^n\right)$

*Note for the last step, in interval 2, cosine is even, so $\cos(-\theta) = \cos\theta$, and for interval 4, $\cos n\pi = (-1)^n$.

$$b_n = \frac{-5}{n\pi}\left(\cos\frac{2n\pi}{3} - 1\right) + \frac{4}{n\pi}\left(\cos\frac{2n\pi}{3} - (-1)^n\right)$$

(1) Factor out $\frac{1}{n\pi}$

$$= \frac{1}{n\pi}\left(-5\left(\cos\frac{2n\pi}{3} - 1\right) + 4\left(\cos\frac{2n\pi}{3} - (-1)^n\right)\right)$$

(2) Distribute coefficients

$$= \frac{1}{n\pi}\left(-5\cos\frac{2n\pi}{3} + 5 + 4\cos\frac{2n\pi}{3} - 4(-1)^n\right)$$

(3) Combine like terms:

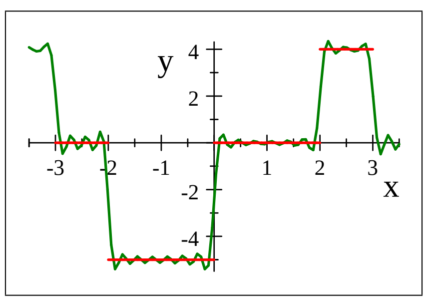
$$b_n = \boxed{\frac{1}{n\pi}\left(-\cos\frac{2n\pi}{3} + 5 - 4(-1)^n\right)}$$

So...

$a_0 = \frac{-2}{3}$ $\quad a_n = -\frac{9}{n\pi}\sin\frac{2n\pi}{3}$ $\quad b_n = \frac{1}{n\pi}\left(-\cos\frac{2n\pi}{3} + 5 - 4(-1)^n\right)$

So plugging into the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right)$ you get

$$f(x) = \frac{-2/3}{2} + \sum_{n=1}^{\infty}\left(\left(-\frac{9}{n\pi}\sin\frac{2n\pi}{3}\right)\cos\frac{n\pi x}{3} + \frac{1}{n\pi}\left(-\cos\frac{2n\pi}{3} + 5 - 4(-1)^n\right)\sin\frac{n\pi x}{3}\right)$$



3 Case 2: $f(x)$ is linear $f(x) = k + mx$

Now we will consider the case were $f(x)$ is linear. In the general case, we will say $f(x) = k + mx$ on $(-p, p)$.

Again, we will consider a generic example to derive "easier" to use formulas. And again, since the function may be broken up within the period, we will derive the formulas using the interval (a, b).

3.1 Calculating a_0

So $a_o = \frac{1}{p}\int_a^b f(x)dx$. Since we are using $f(x) = k + mx$, we get

$$a_0 = \frac{1}{p}\int_a^b (k + mx)\, dx = \frac{1}{p}\left(kx + \frac{m}{2}x^2\right)_{x=a}^{x=b}$$

$$a_0 = \frac{1}{p}\left(k(b-a) + \frac{m}{2}\left(b^2 - a^2\right)\right)$$

3.2 Calculating a_n

As before, I am going to ignore the bounds for now, as well as the $\frac{1}{p}$, which I will put in at the end.

$$a_n = \frac{1}{p}\int_a^b f(x)\cos\frac{n\pi x}{p}dx$$

Substituting in $f(x) = k + mx$ we get (as well as ignoring the bounds and $1/p$)

$$\int (k + mx)\cos\frac{n\pi x}{p}dx$$

Note the mixture of an algebraic with a trigonometric function. This means integration by parts:

$u = k + mx$		$dv = \cos\frac{n\pi x}{p}$
	$\searrow$	
$du = mdx$	$\longleftarrow$	$v = \frac{p}{n\pi}\sin\frac{n\pi x}{p}$

**u-sub for integrating dv

$$= (k + mx)\left(\frac{p}{n\pi}\sin\frac{n\pi x}{p}\right) - \int (m)\left(\frac{p}{n\pi}\sin\frac{n\pi x}{p}\right)dx$$

$$= \frac{p(k+mx)}{n\pi}\sin\frac{n\pi x}{p} - \frac{mp}{n\pi}\int \sin\frac{n\pi x}{p}dx$$ [to do this integration, use u-sub as we have done before]

$$= \frac{p(k+mx)}{n\pi}\sin\frac{n\pi x}{p} - \frac{mp}{n\pi}\left(\frac{p}{n\pi}\right)\left(-\cos\frac{n\pi x}{p}\right)$$

$$= \frac{p(k+mx)}{n\pi}\sin\frac{n\pi x}{p} + \frac{mp^2}{n^2\pi^2}\cos\frac{n\pi x}{p}$$

Now let's multiply by the $\frac{1}{p}$ we took off at the beginning and distribute.

$$\frac{1}{p}\left(\frac{p(k+mx)}{n\pi}\sin\frac{n\pi x}{p} + \frac{mp^2}{n^2\pi^2}\cos\frac{n\pi x}{p}\right) = \frac{k+mx}{n\pi}\sin\frac{n\pi x}{p} + \frac{mp}{n^2\pi^2}\cos\frac{n\pi x}{p}$$

And now evaluate from a to b:

$$a_n = \frac{k+mb}{n\pi}\sin\frac{bn\pi}{p} - \frac{k+ma}{n\pi}\sin\frac{an\pi}{p} + \frac{mp}{n^2\pi^2}\left(\cos\frac{bn\pi}{p} - \cos\frac{an\pi}{p}\right)$$

3.3 Calculating b_n

$$b_n = \frac{1}{p}\int_a^b f(x)\sin\frac{n\pi x}{p}dx$$

I am going to follow the same procedure as above with

$$\int (k+mx)\sin\frac{n\pi x}{p}dx \qquad \text{solve by integration by parts:}$$

$u = k + mx$		$dv = \sin\frac{n\pi x}{p}$
	$\searrow$	
$du = mdx$	$\longleftarrow$	$v = -\frac{p}{n\pi}\cos\frac{n\pi x}{p}$

$$= (k+mx)\left(-\frac{p}{n\pi}\cos\frac{n\pi x}{p}\right) - \int (m)\left(-\frac{p}{n\pi}\cos\frac{\pi n x}{p}\right)dx$$

$$= -\frac{p\,(k+mx)}{n\pi}\cos\frac{n\pi x}{p} + \frac{mp}{n\pi}\int\cos\frac{n\pi x}{p}dx$$

$$= -\frac{p\,(k+mx)}{n\pi}\cos\frac{n\pi x}{p} + \frac{mp}{n\pi}\left(\frac{p}{n\pi}\right)\sin\frac{n\pi x}{p}$$

$$= -\frac{p\,(k+mx)}{n\pi}\cos\frac{n\pi x}{p} + \frac{mp^2}{n^2\pi^2}\sin\frac{n\pi x}{p}$$

Multiply through by $1/p$

$$\frac{1}{p}\left(-\frac{p\,(k+mx)}{n\pi}\cos\frac{n\pi x}{p} + \frac{mp^2}{n^2\pi^2}\sin\frac{n\pi x}{p}\right)$$

$$= -\frac{k+mx}{n\pi}\cos\frac{n\pi x}{p} + \frac{mp}{n^2\pi^2}\sin\frac{n\pi x}{p}$$

And lastly evaluate from a to b.

$$-\frac{k+mx}{n\pi}\left(\cos\frac{bn\pi}{p} - \cos\frac{an\pi}{p}\right) + \frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

$$= \frac{k+ma}{n\pi}\cos\frac{an\pi}{p} - \frac{k+mb}{n\pi}\cos\frac{bn\pi}{p} + \frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

3.4 Summary

If $f(x)$ is linear in the form $f(x) = k + mx$, the coefficients of the Fourier series can be calculated as

$$a_0 = \frac{1}{p}\left(k(b-a) + \frac{m}{2}\left(b^2 - a^2\right)\right)$$

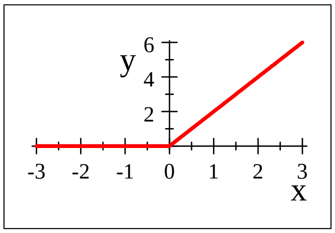
$$a_n = \frac{k+mb}{n\pi}\sin\frac{bn\pi}{p} - \frac{k+ma}{n\pi}\sin\frac{an\pi}{p} + \frac{mp}{n^2\pi^2}\left(\cos\frac{bn\pi}{p} - \cos\frac{an\pi}{p}\right)$$

$$b_n = \frac{k+ma}{n\pi}\cos\frac{an\pi}{p} - \frac{k+mb}{n\pi}\cos\frac{bn\pi}{p} + \frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

(OK, not as easy as the previous kind of problems...)

3.5 Examples

Example 4 $f(x) = \begin{cases} 0 & -3 < x < 0 \\ 2x & 0 < x < 3 \end{cases}$



Not that on the first interval $(-3, 0)$ that $f(x) = 0$, so we only need to concentrate on the second.

So in the interval $(0, 3)\,, p = 3, k = 0, m = 2, a = 0, b = 3$

- Finding a_0 :

$$a_0 = \frac{1}{p}\left(k(b-a) + \frac{m}{2}\left(b^2 - a^2\right)\right)$$

$$a_0 = \frac{1}{3}\left(0(3-0+\frac{2}{2}\left(3^2-0^2\right)\right) = \frac{1}{3}\left(0+1\left(9\right)\right) = \boxed{3}$$

- Finding a_n :

$$a_n = \frac{k+mb}{n\pi}\sin\frac{bn\pi}{p} - \frac{k+ma}{n\pi}\sin\frac{an\pi}{p} + \frac{mp}{n^2\pi^2}\left(\cos\frac{bn\pi}{p} - \cos\frac{an\pi}{p}\right)$$

$$= \frac{0+2(3)}{3}\sin\frac{3n\pi}{3} - \frac{0+2(0)}{3}\sin 0 + \frac{(2)(3)}{n^2\pi^2}\left(\cos\frac{3n\pi}{3} - \cos 0\right)$$

$$= 2\sin n\pi - 0 + \frac{6}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

$$= 0 + \frac{6}{n^2\pi^2}\left(\left(-1\right)^n - 1\right)$$

$$= \boxed{\frac{6}{n^2\pi^2}\left(\left(-1\right)^n - 1\right)}$$

- Finding b_n :

$$b_n = \frac{k+ma}{n\pi}\cos\frac{an\pi}{p} - \frac{k+mb}{n\pi}\cos\frac{bn\pi}{p} + \frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$$

$$= \frac{0+2(0)}{n\pi}\cos 0 - \frac{0+2(3)}{n\pi}\cos\frac{3n\pi}{3} + \frac{2(3)}{n^2\pi^2}\left(\sin\frac{3n\pi}{3} - \sin 0\right)$$

$$= 0 - \frac{6}{n\pi}\cos n\pi + \frac{6}{n^2\pi^2}\left(\sin n\pi\right)$$

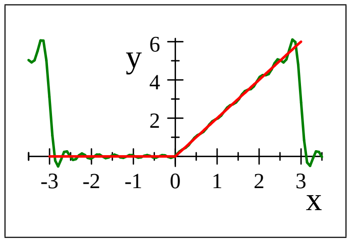
$$= -\frac{6}{n\pi}\left(-1\right)^n + 0$$

$$= \boxed{-\frac{6}{n\pi}\left(-1\right)^n}$$

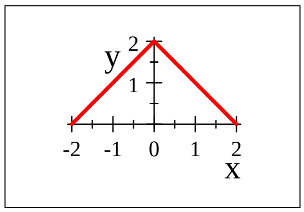
So...$a_0 = 3 \qquad a_n = \frac{6}{n^2\pi^2}\left(\left(-1\right)^n - 1\right) \qquad b_n = \frac{-6}{n\pi}\left(-1\right)^n$

Plugging the coefficients in the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{p} + b_n \sin \frac{n\pi x}{p} \right)$ you get

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \left(\left(\frac{6}{n^2\pi^2} \left((-1)^n - 1\right) \right) \cos \frac{n\pi x}{3} + \left(\frac{-6}{n\pi} (-1)^n \right) \sin \frac{n\pi x}{3} \right)$$



Example 5 $f(x) = \begin{cases} x+2 & -2 < x < 0 \\ 2-x & 0 < x < 2 \end{cases}$



Now we have two intervals to worry about:

Interval 1 $(-2, 0) : p = 2, k = 2, m = 1, a = -2, b = 0$
Interval 2: $(0, 2) : p = 2, k = 2, m = -1, a = 0, b = 2$

- Finding a_0 : $\frac{1}{p} \left(k(b-a) + \frac{m}{2} \left(b^2 - a^2\right) \right)$

Interval 1: $\frac{1}{2} (2(0-(-2)) + \frac{1}{2} \left(0^2 - (-2)^2\right) = \frac{1}{2} (4-2) = 1$

Interval 2: $\frac{1}{2} \left(2(2-0) + \frac{-1}{2} \left(2^2 - 0^2\right) \right) = \frac{1}{2} (4-2) = 1$

So $a_0 = 1 + 1 = \boxed{2}$

- Finding a_n : $\frac{k+mb}{n\pi} \sin \frac{bn\pi}{p} - \frac{k+ma}{n\pi} \sin \frac{an\pi}{p} + \frac{mp}{n^2\pi^2} \left(\cos \frac{bn\pi}{p} - \cos \frac{an\pi}{p} \right)$

Interval 1: $\frac{2+1(0)}{n\pi} \sin 0 - \frac{2+1(-2)}{n\pi} \sin \frac{2n\pi}{2} + \frac{1(2)}{n^2\pi^2} \left(\cos 0 - \cos \frac{-2n\pi}{2} \right)$

$$= 0 - 0 + \frac{2}{n^2\pi^2} (1 - \cos n\pi))$$

$$= \frac{2}{n^2\pi^2} (1 - (-1)^n)$$

Interval 2: $\frac{2-1(2)}{n\pi} \sin \frac{2n\pi}{2} - \frac{2-1(0)}{n\pi} \sin 0 + \frac{-1(2)}{n^2\pi^2} \left(\cos \frac{2n\pi}{2} - \cos 0 \right)$

$$= 0 - 0 - \frac{2}{n^2\pi^2}(\cos n\pi - 1)$$
$$= -\frac{2}{n^2\pi^2}((-1)^n - 1)$$
$$= \frac{2}{n^2\pi^2}(1 - (-1)^n)$$

Adding them together gives

$$a_n = \frac{2}{n^2\pi^2}(1 - (-1)^n) + \frac{2}{n^2\pi^2}(1 - (-1)^n)$$
$$a_n = \boxed{\frac{4}{n^2\pi^2}(1 - (-1)^n)}$$

- Finding b_n : $\frac{k+ma}{n\pi}\cos\frac{an\pi}{p} - \frac{k+mb}{n\pi}\cos\frac{bn\pi}{p} + \frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$

Interval 1: $\frac{2+1(-2)}{n\pi}\cos\frac{-2n\pi}{2} - \frac{2+1(0)}{n\pi}\cos 0 + \frac{1(2)}{n^2\pi^2}\left(\sin 0 - \sin\frac{-2n\pi}{2}\right)$

$$= 0 - \frac{2}{n\pi} - \frac{2}{n^2\pi^2}(0 - 0)$$
$$= -\frac{2}{n\pi}$$

Interval 2: $\frac{2-1(0)}{n\pi}\cos 0 - \frac{2-1(2)}{n\pi}\cos\frac{2n\pi}{2} + \frac{-1(2)}{n^2\pi^2}\left(\sin\frac{2n\pi}{2} - \sin 0\right)$

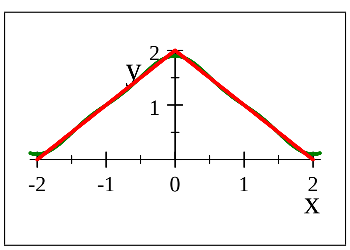
$$= \frac{2}{n\pi} - 0 - 0$$
$$= \frac{2}{n\pi}$$

So $b_n = -\frac{2}{n\pi} + \frac{2}{n\pi} = \boxed{0}$

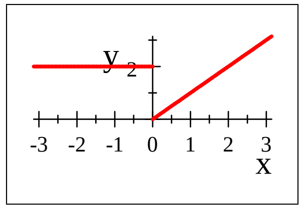
Then we have $a_0 = 2$ $\quad a_n = \frac{4}{n^2\pi^2}(1 - (-1)^n)$ $\quad b_n = 0$

plugging the coefficients in the Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right)$ you get

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty}\left(\left(\frac{4}{n^2\pi^2}(1 - (-1)^n)\right)\cos\frac{n\pi x}{2} + (0)\sin\frac{n\pi x}{2}\right)$$
$$f(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}(1 - (-1)^n)\right)\cos\frac{n\pi x}{2}$$



Example 6 $f(x) = \begin{cases} 2 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$



Note this one that it is a mixture of both constant and linear functions. Therefore, on the lower interval, we can use the formulas for a constant function, and for the linear, the second set of formulas:

Interval 1: $(-\pi, 0) : p = \pi, k = 2, a = -\pi, b = 0$
Interval 2: $(0, \pi) : p = \pi, k = 0, m = 1, a = 0, b = \pi$

- Finding a_0 :

Interval 1: $\frac{k}{p}(b - a)$

$$\frac{2}{\pi}(0 - (-\pi)) = \frac{2\pi}{\pi} = 2$$

Interval 2: $\frac{1}{p}\left(k(b-a) + \frac{m}{2}\left(b^2 - a^2\right)\right)$

$$\frac{1}{\pi}\left(0 + \frac{1}{2}\left(\pi^2 - 0^2\right)\right) = \frac{1}{\pi}\left(-\frac{\pi^2}{2}\right) = -\frac{\pi}{2}$$

So $a_0 = \boxed{2 - \frac{\pi}{2}}$

- Finding a_n :

Interval 1: $\frac{k}{n\pi}\left(\sin\frac{bn\pi}{p} - \sin\frac{an\pi}{p}\right)$

$$\frac{2}{n\pi}\left(\sin 0 - \sin\frac{-\pi n\pi}{\pi}\right) = \frac{1}{n\pi}\left(0 - \sin\left(-n\pi\right)\right) = 0$$

Interval 2: $\frac{k+mb}{n\pi}\sin\frac{bn\pi}{p} - \frac{k+ma}{n\pi}\sin\frac{an\pi}{p} + \frac{mp}{n^2\pi^2}\left(\cos\frac{bn\pi}{p} - \cos\frac{an\pi}{p}\right)$

$$= \frac{0+1\,(\pi)}{\pi}\sin\frac{\pi n\pi}{\pi} - \frac{0+1(0)}{\pi}\sin 0 + \frac{1(\pi)}{n^2\pi^2}\left(\cos\frac{\pi n\pi}{\pi} - \cos 0\right)$$

$$= \sin n\pi - 0 + \frac{1}{n^2\pi}\left(\cos n\pi - 1\right)$$

$$= \frac{1}{n^2\pi}\left((-1)^n - 1\right)$$

So $a_n = 0 + \frac{1}{n^2\pi}\left((-1)^n - 1\right) = \boxed{\frac{1}{n^2\pi}\left((-1)^n - 1\right)}$

- Finding b_n :

Interval 1: $\frac{k}{n\pi}\left(\cos\frac{an\pi}{p}-\cos\frac{bn\pi}{p}\right)$

$=\frac{2}{n\pi}\left(\cos\frac{-\pi n\pi}{\pi}-\cos 0\right)$

$=\frac{2}{n\pi}\left(\cos\left(-n\pi\right)-1\right)$

$=\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)$

Interval 2: $\frac{k+ma}{n\pi}\cos\frac{an\pi}{p}-\frac{k+mb}{n\pi}\cos\frac{bn\pi}{p}+\frac{mp}{n^2\pi^2}\left(\sin\frac{bn\pi}{p}-\sin\frac{an\pi}{p}\right)$

$=\frac{0+1(0)}{n\pi}\cos 0-\frac{0+1(\pi)}{n\pi}\cos\frac{\pi n\pi}{\pi}+\frac{1(\pi)}{n^2\pi^2}\left(\sin\frac{\pi n\pi}{\pi}-\sin 0\right)$

$=0-\frac{1}{n}\cos n\pi+0$

$=-\frac{1}{n}\left(-1\right)^n$

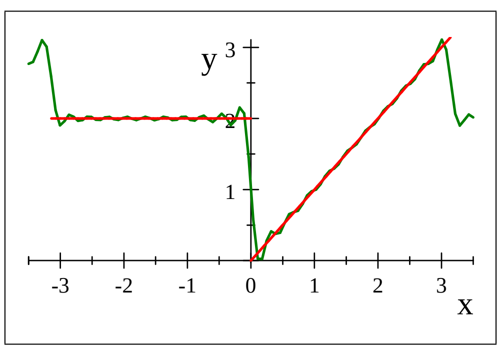
So $b_n = \boxed{\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)-\frac{1}{n}\left(-1\right)^n}$

Which gives: $a_0=2+\frac{\pi}{2}$ $\quad a_n=\frac{1}{n^2\pi}\left(\left(-1\right)^n-1\right)$ $\quad b_n=\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)-\frac{1}{n}\left(-1\right)^n$

plugging the coefficients in the Fourier series $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{p}+b_n\sin\frac{n\pi x}{p}\right)$ you get

$$f(x)=\frac{2-\frac{\pi}{2}}{2}+\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2\pi}\left(\left(-1\right)^n-1\right)\right)\cos\frac{n\pi x}{\pi}+\left(\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)-\frac{1}{n}\left(-1\right)^n\right)\sin\frac{n\pi x}{\pi}\right)$$

$$f(x)=\frac{4+\pi}{4}+\sum_{n=1}^{\infty}\left(\left(\frac{1}{n^2\pi}\left(\left(-1\right)^n-1\right)\right)\cos nx+\left(\frac{2}{n\pi}\left(\left(-1\right)^n-1\right)-\frac{1}{n}\left(-1\right)^n\right)\sin nx\right)$$



4 Convergence at Points of Discontinuity

One advantage that a Fourier series gives is that it uses a continuous function to describe a function that might have discontinuities. Sometimes it might be useful to find the value to which the Fourier series converges at points where there is a jump discontinuity within one period. This can easily be found by taking the average value of the function at both sides of the point of discontinuity.

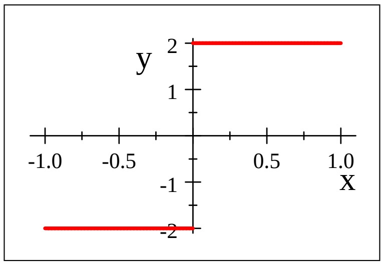
If a function has a discontinuity at x_0, the value to which the Fourier series converges at that point is

$$F_0 = \frac{f(x_0^-) + f(x_0^+)}{2}$$

where $f(x_0^-)$ is the value of $f(x)$ on the left side, and $f(x_0^+)$ is the value of $f(x)$ on the right side.

Let's look at the previous examples:

Example 1 $f(x) = \begin{cases} -2 & -1 < x < 0 \\ 2 & 0 < x < 1 \end{cases}$

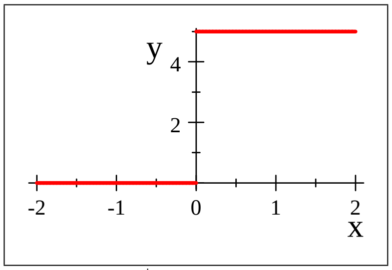


This function is discontinuous at $x_0 = 0$. On the left side, it has a value of $f(x_0^-) = -2$, and on the right side $f(x_0^+) = 2$

$$F_0 = \frac{= 2 + 2}{2} = 0$$

So the Fourier series converges to $(x_0, F_0) = (0, 0)$

Example 2 $f(x) = \begin{cases} 0 & -2 < x < 0 \\ 5 & 0 < x < 2 \end{cases}$

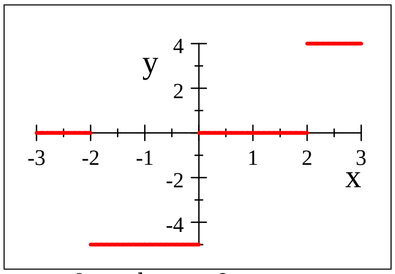


The point of discontinuity occurs at $x_0 = 0$, with $f(x_0^-) = 0$ and $f(x_0^+) = 5$

$$F_0 = \frac{0 + 5}{2} = \frac{5}{2}$$

So at the point of discontinuity, the Fourier series converges to $(x_0, F_0) = \left(0, \frac{5}{2}\right)$

Example 3 $f(x) = \begin{cases} 0 & -3 < x < -2 \\ -5 & -2 < x < 0 \\ 0 & 0 < x < 2 \\ 4 & 2 < x < 3 \end{cases}$



This function has 3 points of discontinuity and $x_0 = -2, x_0 = 0$, and $x_0 = 2$.

At $x_0 = -2:$

$f(x_0^-) = 0$ and $f(x_0^+) = -5 \quad \rightarrow \quad F_0 = \frac{0 + (-5)}{2} = -\frac{5}{2}$

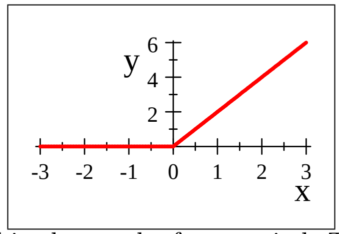
At $x_0 = 0:$

$f(x_0^-) = -5$ and $f(x_0^+) = 0 \quad \rightarrow \quad F_0 = \frac{-5 + 0}{2} = -\frac{5}{2}$

At $x_0 = 2:$
$f(x_0^-) = 0$ and $f(x_0^+) = 4 \qquad \rightarrow \qquad F_0 = \dfrac{0+4}{2} = 2$

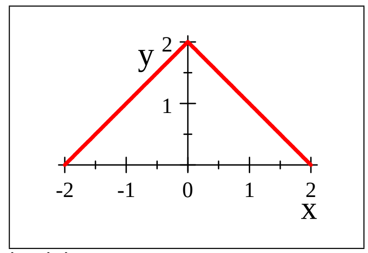
So the points of convergence (x_0, F_0) would be $\left(-2, -\frac{5}{2}\right), \left(0, -\frac{5}{2}\right)$ and $(2, 2)$

Example 4 $f(x) = \begin{cases} 0 & -3 < x < 0 \\ 2x & 0 < x < 3 \end{cases}$



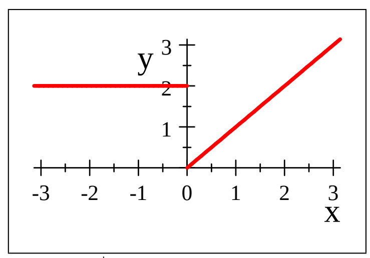
In this function, there are no jump discontinuities within the graph of one period. Therefore, there would be no points of convergence for discontinuities.

Example 5 $f(x) = \begin{cases} x+2 & -2 < x < 0 \\ 2-x & 0 < x < 2 \end{cases}$



Like the previous example, this function has no jump discontinuities.

Example 6 $f(x) = \begin{cases} 2 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$



The function is discontinuous at $x_0 = 0$, with $f(x_0^-) = 2$ and $f(x_0^+) = 0$, so

$$F_0 = \frac{2+0}{2} = 1$$

Giving $(x_0, F_0) = (0, 1)$