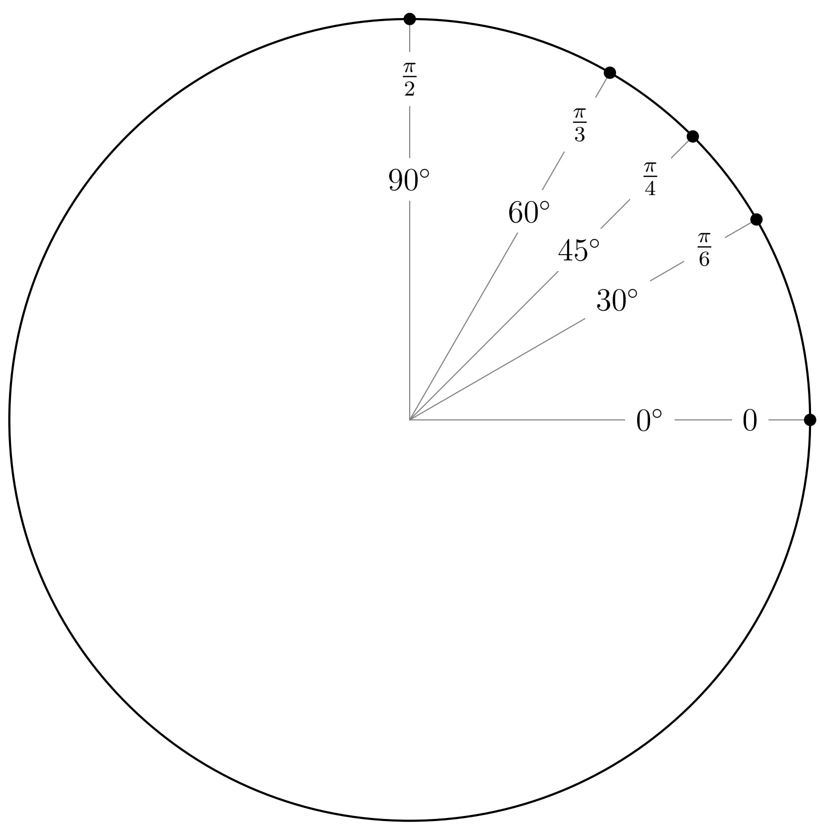
This document is intended to help you reproduce the unit circle on the fly without having to memorize loads and loads of trig values. Here is a summary of the steps we will perform:

- Step 1: Construct the unit circle for $\sin\theta$, where θ ranges between 0 and $\frac{\pi}{2}$ radians.
- Step 2: Complete the unit circle for $\sin\theta$ and values of θ between $\frac{\pi}{2}$ and π.
- Step 3: Fill out the unit circle for $\sin\theta$ by using a simple flip of steps 1 and 2.
- Step 4: Complete the unit circle for $\cos\theta$ by rotating your $\sin\theta$ circle by 90 degrees.

You can then use the unit circle for $\sin\theta$ and $\cos\theta$ to derive, for example, values of $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

Before starting, recall that the x-coordinate on the unit circle corresponds to $\cos\theta$, and the y-coordinate represents $\sin\theta$. So, because we're first filling out the sine values, we're actually filling out all of the y-coordinates before we fill out the x-coordinates.

- Step 1: Start out by drawing a circle, and labelling the five angles in the first quadrant, in increasing order: these values (in order, measured in radians) are $\theta = 0,\ \frac{\pi}{6},\ \frac{\pi}{4}, \frac{\pi}{3}$, and $\frac{\pi}{2}$. The equivalent values in degrees are $\theta = 0,\ 30,\ 45,\ 60,\ 90$.



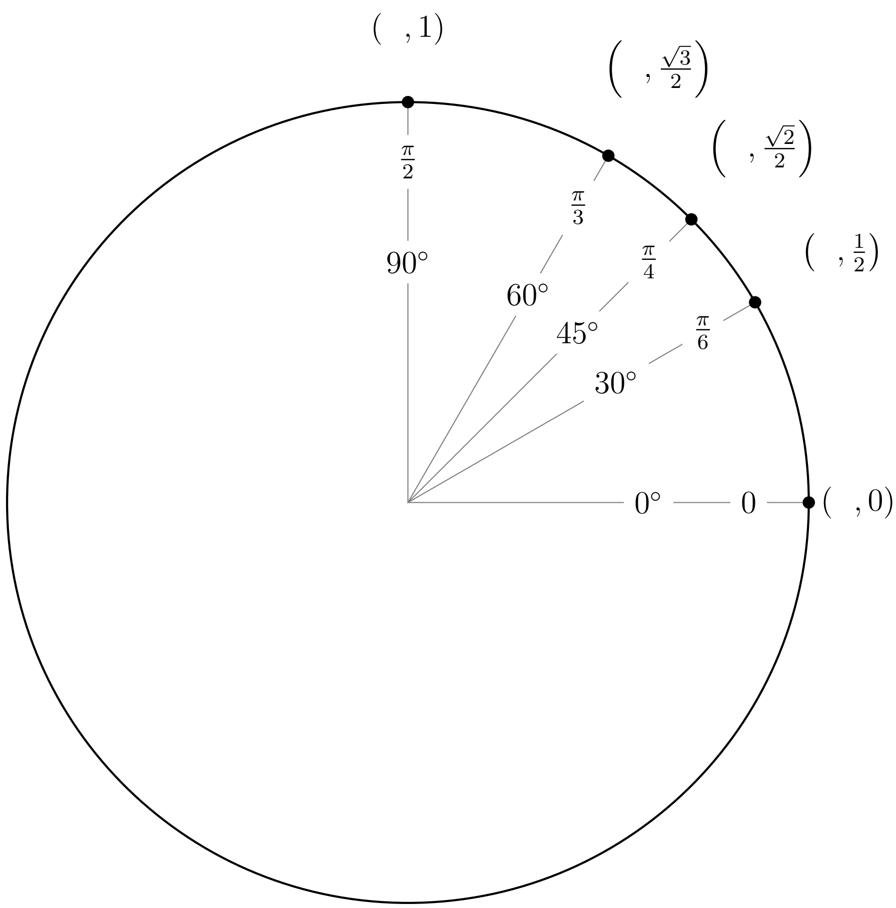
The values of $\sin\theta$ in the first quadrant have a nice symmetry about them if you write them in this form:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	$\frac{\sqrt{0}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$

When written in this fashion, all of them have a 2 in the denominator, a square root in the numerator, and the numbers inside the roots just count up monotonically from 0 to 4. Of course, some of these simplify: $\sqrt{0} = 0$, $\sqrt{1} = 1$, $\sqrt{4} = 2$, so the sine values reduce to the following:

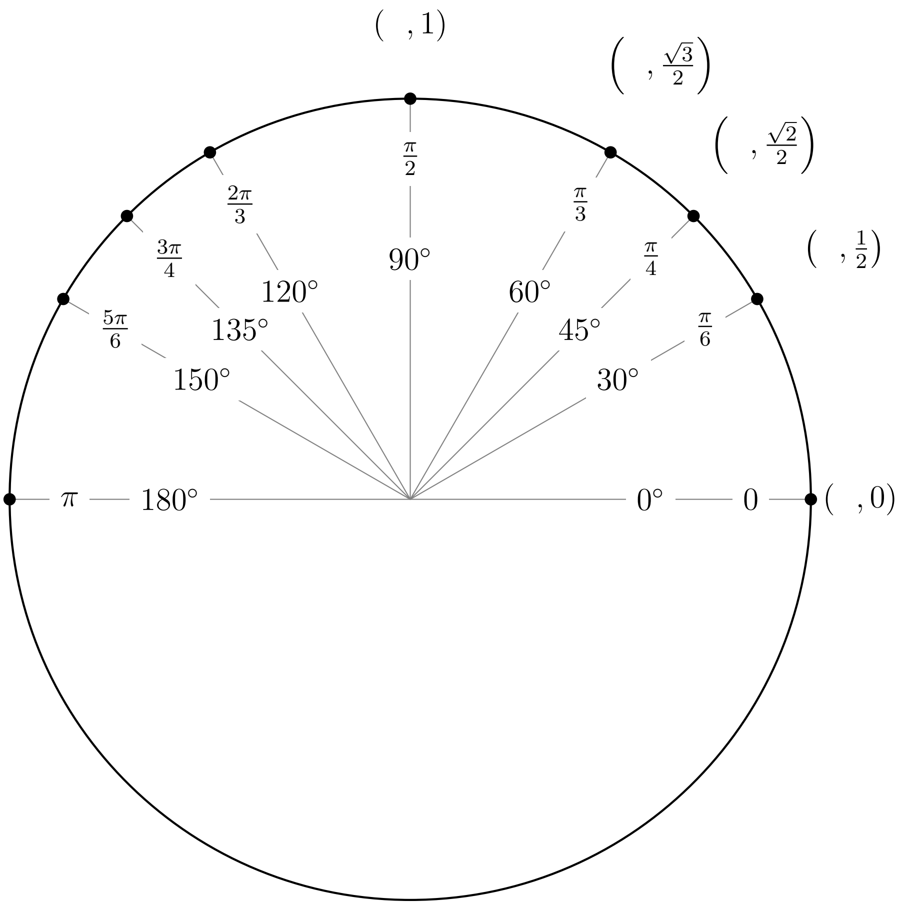
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Let's include these in the circle as the second coordinate:



- Step 2: Now we include the values of $\sin\theta$ which are between $\frac{\pi}{2}$ and π radians. To get the angle measurements, add $\frac{\pi}{2}$ (or 90 degrees) to each of the angles in the first quadrant, i.e. the angles are

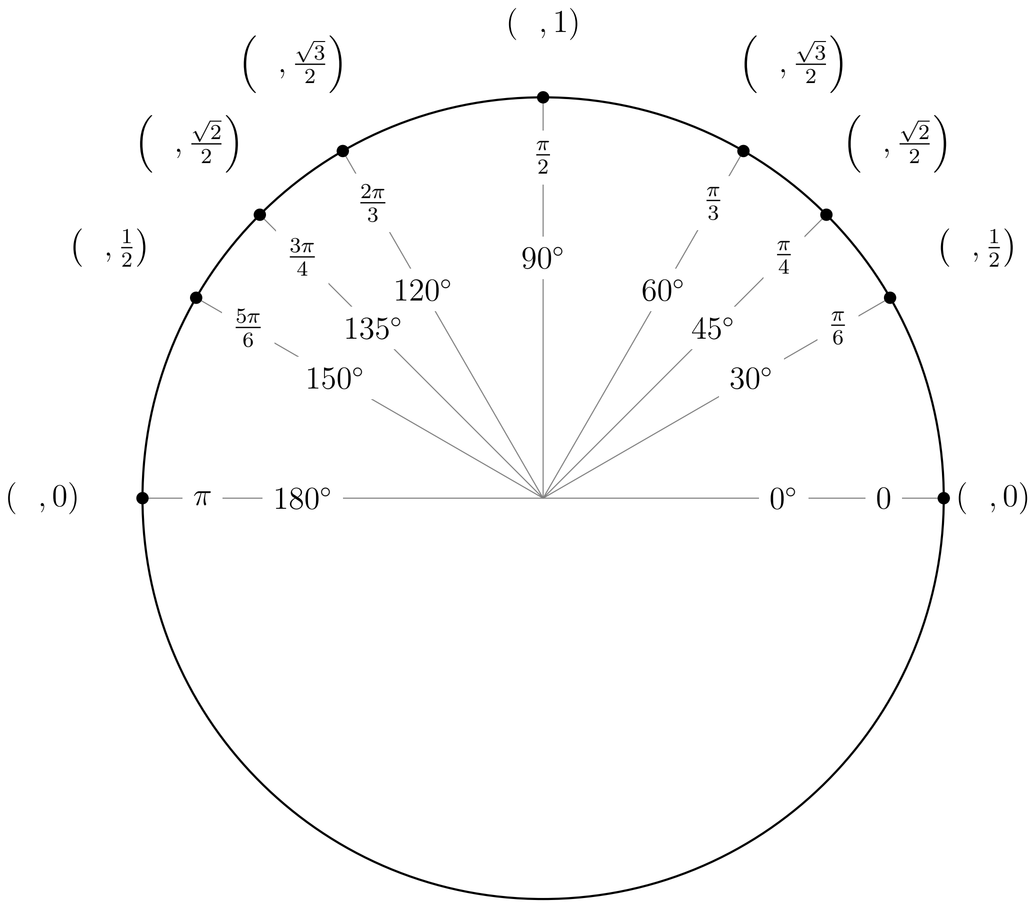
$$0+\frac{\pi}{2}=\frac{\pi}{2}, \qquad \frac{\pi}{6}+\frac{\pi}{2}=\frac{4\pi}{6}=\frac{2\pi}{3}, \qquad \frac{\pi}{4}+\frac{\pi}{2}=\frac{3\pi}{4}, \qquad \frac{\pi}{3}+\frac{\pi}{2}=\frac{5\pi}{6}, \qquad \frac{\pi}{2}+\frac{\pi}{2}=\pi.$$



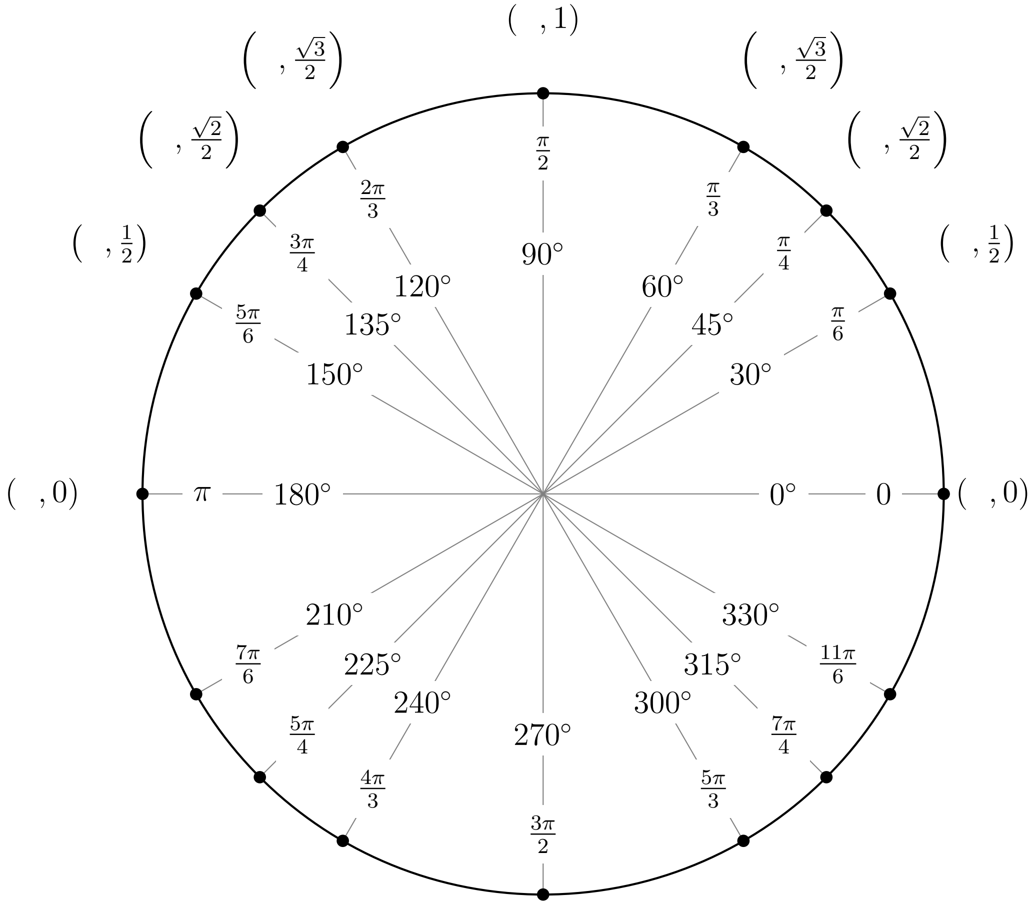
In order to fill in the rest of the sine values between $\frac{\pi}{2}$ and π, you just reverse the order of the values between 0 and $\frac{\pi}{2}$. That is, count downward from 1 to 0 (instead of upward from 0 to 1): this results in the following:

θ	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

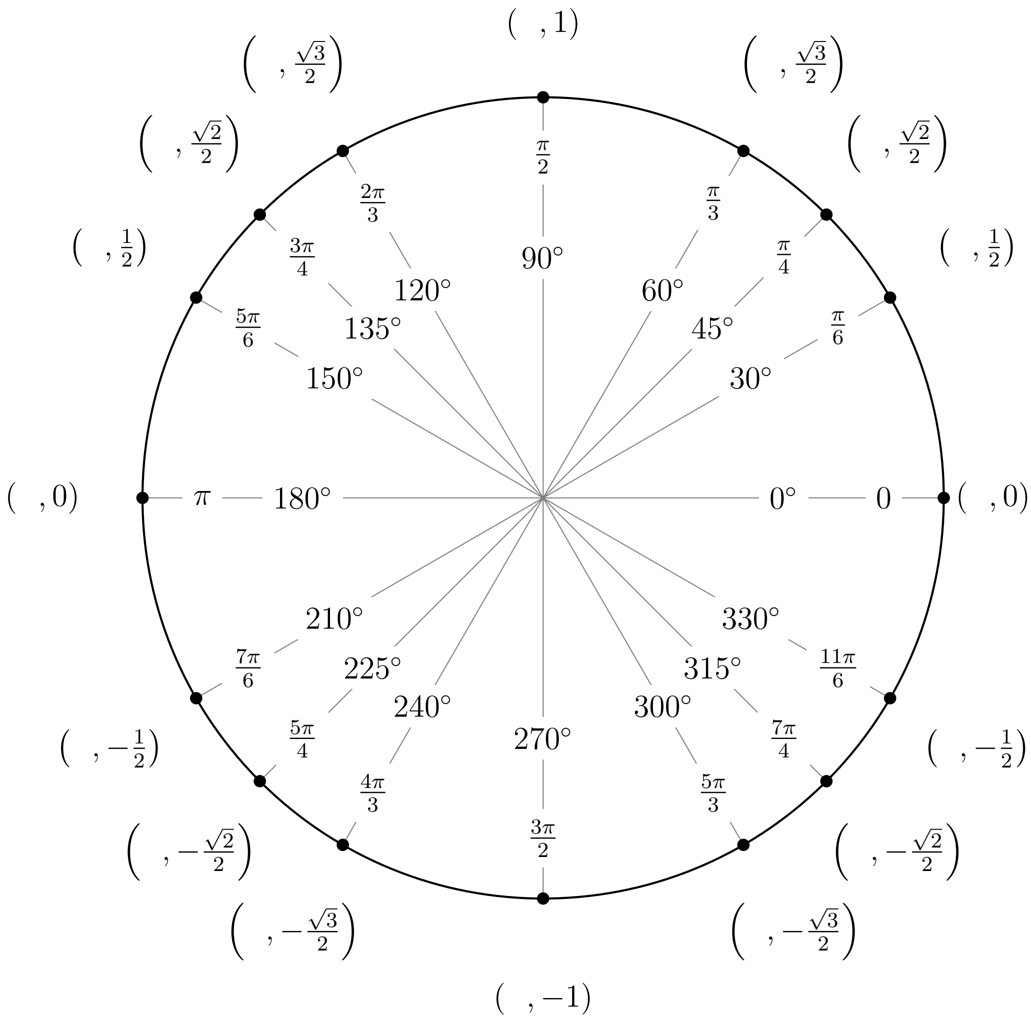
Placing these on the graph, again in the second coordinate, gives the following:



- Step 3: Let's finish up the sine coordinates for θ between π and 2π. To get the angles, just add π radians to each angle starting from zero (i.e. $0 + \pi$, $\frac{\pi}{6} + \pi = \frac{7\pi}{6}$, $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$, etc.) while going around counterclockwise:



In order to get the y-coordinates on the bottom half of the unit circle, you can just take the y-coordinate of the point directly above the point and negate it. For example, at $\theta = \frac{5\pi}{3}$, the point directly above it corresponds to the angle $\frac{\pi}{3}$ which has y-coordinate $\frac{\sqrt{3}}{2}$. Thus the y-coordinate at $\theta = \frac{5\pi}{3}$ is equal to $-\frac{\sqrt{3}}{2}$. The bottom line is that this step amounts to flipping the circle upside down to get the lower half and negating each of the y-coordinates. Filling these in gives the following:

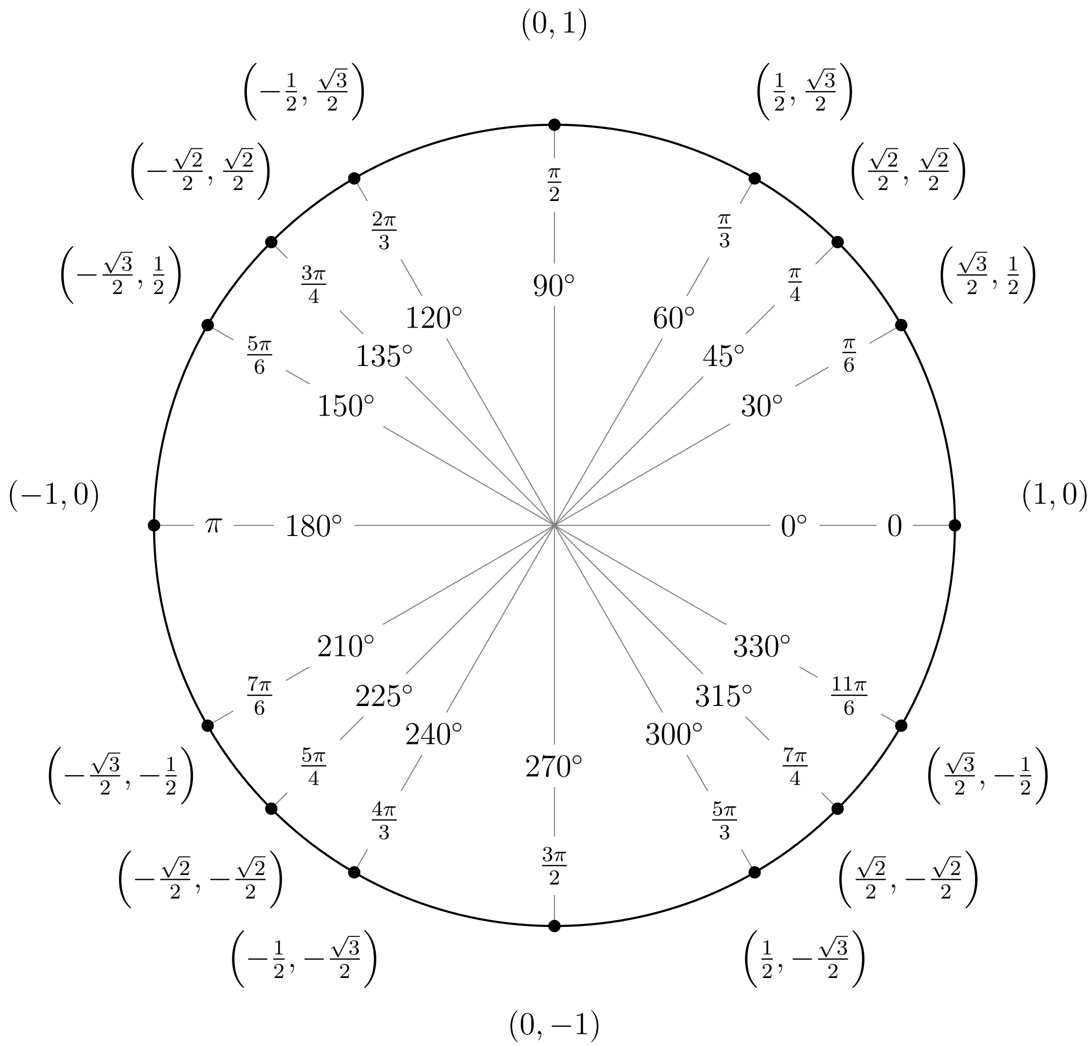


- Step 4: Finally, we're ready to complete the x-coordinates, which correspond to the values of $\cos\theta$. This part is now really easy, as all of the hard work has been done! The key is the following statement:

  The cosine of an angle corresponds to the sine of that angle plus $\frac{\pi}{2}$.

  So, for example, the cosine of $\frac{\pi}{2}$ is the exact same as the sine of $\frac{\pi}{2}+\frac{\pi}{2}=\pi$; hence $\cos\frac{\pi}{2}=\sin\pi=0$. Similarly, the cosine of $\frac{5\pi}{4}$ is equal to the sine of $\frac{5\pi}{4}+\frac{\pi}{2}=\frac{7\pi}{4}$; therefore $\cos\frac{5\pi}{4}=\sin\frac{7\pi}{4}=-\frac{\sqrt{2}}{2}$. In terms of how to fill this in on the unit circle, start from an angle θ. Move 90 degrees counterclockwise, and write the sine value corresponding to that new angle into the value of cosine at your starting θ. For example, if you start at $\theta = \frac{4\pi}{3}$, rotating counterclockwise by 90 degrees lands us at $\frac{11\pi}{6}$, for which the sine value is $-\frac{1}{2}$. Therefore, the cosine of our starting angle, $\frac{4\pi}{3}$, is equal to $-\frac{1}{2}$, too.

Below is the final, finished unit circle constructed in this way. Notice that if you start from $\theta = 0$ and read off the values of $\cos\theta$ (the x-coordinates), you get the exact same list as if you had started from $\theta = \frac{\pi}{2}$ and read off the values of $\sin\theta$ (the y-coordinates).



As a final suggestion, see if you can reproduce this on your own a couple of times (no peeking!). If you can, then you should be good to go for the exam!