MA 161 Exam 2 (Green)

1. If $f(t) = \dfrac{3t-2}{(3+\sqrt{t})^2}$, then $f'(4)$ is:

A. 2

B. $-\dfrac{1}{25}$

C. $\dfrac{2}{25}$

D. $-\dfrac{1}{(25)^3}$

E. $-\dfrac{1}{(25)^4}$

$$f' = \frac{u'v - v'u}{v^2} = \frac{(3)(3+\sqrt{t})^2 - 2\cdot(3+t^{1/2})\frac{1}{2t^{1/2}}(3t-2)}{[(3+\sqrt{t})^2]^2}$$

$$f'(4) = \frac{3(3+2)^2 - 2(3+2)\frac{1}{2\cdot 2}\underbrace{(3\cdot 4-2)}_{10}}{(3+2)^4}$$

$$= \frac{3\cdot 5^2 - 5^2}{5^4} = \frac{2}{5^2} = \frac{2}{25}$$

2. Find the derivative of $y = (x^2-3)(7^x)$ at $x=2$. Note that $\dfrac{d}{dx}(a^x) = (\ln a)a^x$

A. 196

B. $196 - 49\ln 7$

C. $196\ln 7$

D. $196 + 49\ln 7$

E. 49

$$y' = u'v + uv' = (2x)7^x + (x^2-3)(\ln 7)7^x$$

$$y'(2) = 2\cdot 2\,7^2 + (4-3)(\ln 7)7^2$$

$$= 4\cdot 49 + (\ln 7)49$$

$$= 196 + 49\ln 7$$

3. Find y'' if $y = \sin(x^2)$.

A. $2\cos(x^2) - 4x^2\sin(x^2)$
B. $\cos(x^2) - \sin(x^2)$
C. $2x\cos(x^2) - 4x^2\sin(x^2)$
D. $2x\cos(x^2) + 2x\sin(x^2)$
E. $-\sin(x^2)$

$y = \text{Sin } u$ where $u = x^2$. So

$$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx} = \text{Cos}\, u\, (2x) = 2x\, \text{Cos}\, x^2$$

$$y'' = u'v + uv' = 2\,\text{Cos}\, x^2 + 2x\left[(-\text{Sin}\, x^2)\, 2x\right]$$

$$= 2\,\text{Cos}\, x^2 - 4x^2\, \text{Sin}\, x^2$$

4. Find the limit: $\displaystyle\lim_{x\to 0} \frac{\tan^2(2x)}{\sin^2(3x)}$

A. 9/4
B. 2/3
C. 4/9
D. 3/2
E. 1

$$\frac{\text{Tan}^2(2x)}{\text{Sin}^2(3x)} = \frac{1}{\text{Sin}^2(3x)}\,\frac{\text{Sin}^2(2x)}{\text{Cos}^2(2x)}\cdot\frac{(3x)^2}{(3x)^2}\cdot\frac{(2x)^2}{(2x)^2}$$

$$= \left[\frac{(3x)}{\text{Sin}(3x)}\right]^2\left[\frac{\text{Sin}(2x)}{2x}\right]^2\frac{1}{\text{Cos}^2(2x)}\cdot\frac{2^2}{3^2}$$

$$\xrightarrow[x\to 0]{} \left[\frac{1}{1}\right]^2\left[1\right]^2\frac{1}{1^2}\cdot\frac{4}{9}$$

$$= \frac{4}{9}$$

5. Find the slope of the line tangent to the curve given by the equation

$$\tan(xy) = 8y^2 - \sin x$$

at a point $(x, y) = (\frac{\pi}{2}, \frac{1}{2})$.

A. $\frac{1}{4}$

B. $\frac{1}{8-\pi}$

C. ∞

D. $\frac{2-\pi}{4+\pi}$

E. $\frac{\pi}{2-\pi}$

$$\sec^2(xy)\frac{d}{dx}(xy) = 8 \cdot 2yy' - \cos x$$

$$\sec^2(xy)\left[1 \cdot y + xy'\right] = 16yy' - \cos x$$

At $(\frac{\pi}{2}, \frac{1}{2})$, $x = \frac{\pi}{2}$, $y = \frac{1}{2}$ and $xy = \frac{\pi}{4}$.

$$\left(\sec^2\frac{\pi}{4}\right)\left[\frac{1}{2} + \frac{\pi}{2}y'\right] = 16 \cdot \frac{1}{2}y' - \underbrace{\cos\frac{\pi}{2}}_{0}$$

$$2\left[\frac{1}{2} + \frac{\pi}{2}y'\right] = 8y' - 0$$

$$1 + \pi y' = 8y' \qquad y' = \frac{1}{8-\pi}$$

$$\sec\frac{\pi}{4} = \frac{\sqrt{2}}{1}$$

6. Find the derivative $\frac{dy}{dx}$ when $y = x^{\sin x}$.

A. $\frac{dy}{dx} = (\sin x) \cdot x^{\sin x - 1}$

B. $\frac{dy}{dx} = (\ln x) \cdot x^{\sin x} \cdot \cos x$

C. $\frac{dy}{dx} = (\sin x) \cdot x^{\sin x - 1} + (\ln x) \cdot x^{\sin x} \cdot \cos x$

D. $\frac{dy}{dx} = x^{\sin x} \cdot (\ln x + \sin x)$

E. $\frac{dy}{dx} = x^{\sin x} \cdot (\cos x + \frac{1}{x})$

$$\ln y = \ln x^{\sin x} = \sin x \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = (\cos x)\ln x + (\sin x)\frac{1}{x}$$

$$y = x^{\sin x}$$

$$\frac{dy}{dx} = x^{\sin x}\left[(\cos x)\ln x + \frac{1}{x}\sin x\right]$$

$$= x^{\sin x}(\cos x)\ln x + x^{\sin x - 1}\sin x$$

$$\frac{x^{\sin x}}{x} = x^{\sin x - 1}$$

7. Find the slope of the line tangent to the graph $y = x(\ln x)^3$ when $x = e$.

A. $\frac{3}{e}$

B. 3

C. $3e + 1$

D. 6

E. 4

$$y' = 1 \cdot (\ln x)^3 + x\left[3(\ln x)^2 \cdot \frac{1}{x}\right]$$
$$= (\ln x)^3 + 3(\ln x)^2$$

At $x = e$: $y' = (\ln e)^3 + 3(\ln e)^2$

$\ln e = 1$ $\quad = 1^3 + 3 \cdot 1^2 = 4$

8. If the radius of a circular ink blot is growing at a rate of 3 cm/min, how fast (in cm^2/min) is the area of the blot growing when the radius is 10 cm?

A. 30π

B. 600π

C. 20π

D. 60π

E. 300π

$$A = \pi r^2$$
$$\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt} = 2\pi r \underbrace{\frac{dr}{dt}}_{3}$$

When $r = 10$: $\frac{dA}{dt} = (2\pi \cdot 10) \cdot 3$

$$= 60\pi$$

9. Which of these is equal to $\sinh x \cosh x$?

A. $\frac{1}{4}\sinh 2x$

B. $\frac{1}{2}\sinh 2x$

C. $\sinh 2x$

D. $2\sinh 2x$

E. $4\sinh 2x$

$$\parallel$$

$$\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{x}+e^{-x}}{2}\right)=$$

$$\frac{1}{2\cdot 2}\left[\underbrace{e^{x}e^{x}}_{e^{2x}}+\underbrace{e^{x}e^{-x}}_{1}-\underbrace{e^{-x}e^{x}}_{1}-\underbrace{e^{-x}e^{-x}}_{e^{-2x}}\right]$$

$$=\frac{1}{2}\left[\frac{e^{2x}-e^{-2x}}{2}\right]=\frac{1}{2}\sinh 2x$$

10. There is 100 grams of a certain radioactive element at noon. At 2:00 PM there is 50 grams. How much will there be at 3:00 PM?

$t=0$ at noon.

A. $100e^{(-3/32)\ln 2}$ grams

B. $100e^{(-3/16)\ln 2}$ grams

C. $100e^{(-3/8)\ln 2}$ grams

D. $100e^{(-3/4)\ln 2}$ grams

E. $100e^{(-3/2)\ln 2}$ grams

$$m(t)=100e^{-kt}$$

$m_0=100$

2 pm: $m(2)=100e^{-k2}=50$. So $e^{-k2}=\frac{1}{2}$

$-k2=\ln\frac{1}{2}$

$=-\ln 2$

So $k=\frac{\ln 2}{2}$

Hence, $m(t)=100e^{-(\ln 2/2)t}$.

Finally, at 3 pm,

$$m(3)=100e^{-3\ln 2/2}$$

11. Let $f(x) = \sinh(\cos x) - \cosh(\sin x)$. Find the exact value of $f'(\pi/2)$.

A. $\dfrac{e - e^{-1}}{2}$

B. -1

C. $\dfrac{e}{2}$

D. $-\dfrac{1}{2}$

E. $\dfrac{e + e^{-1}}{2}$

$$f'(x) = \cosh(\cos x)(-\sin x) - \sinh(\sin x)\cos x$$

$$\sin\frac{\pi}{2} = 1, \quad \cos\frac{\pi}{2} = 0$$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Finally, $f'\left(\frac{\pi}{2}\right) = \cosh(0)\left(-\sin\frac{\pi}{2}\right) - \sinh(1)\left(\cos\frac{\pi}{2}\right) = \left[\frac{1+1}{2}\right]\cdot(-1) - \left[\frac{e-e^{-1}}{2}\right]\cdot 0 = -1$

12. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole along a straight path. The tip of his shadow is moving at the speed of 5 ft/s when he is 40 ft away from the pole. How fast is the man walking at that instant ?

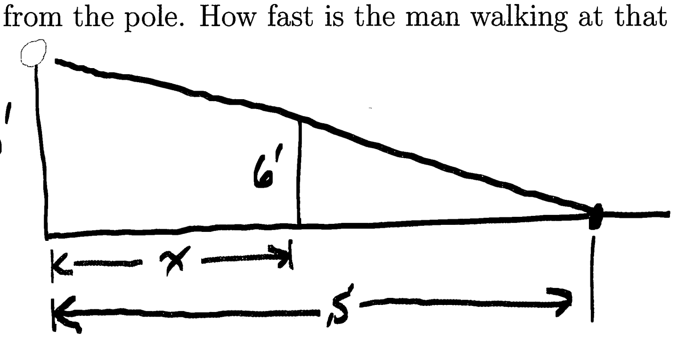
A. 4 ft/s

B. 5 ft/s

C. $\dfrac{3}{40}$ ft/s

D. 3 ft/s

E. $\dfrac{1}{8}$ ft/s



$$\frac{s}{15} = \frac{s - x}{6}$$

$$\frac{x}{6} = s\left(\frac{1}{6} - \frac{1}{15}\right)$$

$$x = s\left(1 - \frac{2}{5}\right)$$

$$\boxed{x = \frac{3}{5}s}$$

Know $\dfrac{ds}{dt} = 5.$

Want $\dfrac{dx}{dt}$

So $\dfrac{dx}{dt} = \dfrac{3}{5}\dfrac{ds}{dt} = \dfrac{3}{5}\cdot 5 = 3$