MA 22000 Lesson 25 Notes
Section 5.3 (2^{nd} half of textbook)

Higher Derivatives:
In this lesson, we will find a derivative of a derivative. A second derivative is a derivative of the first derivative. A third derivative is a derivative of a second derivative, etc.

Notation for Higher Derivatives
For the second derivative of a function, $y = f(x)$, any of the following notations may be used.

$$f''(x), \quad \frac{d^2y}{dx^2}, \quad D_x^2[f(x)], \quad y''$$

For the third derivative of a function, similar notation is used.

$$f'''(x), \quad \frac{d^3y}{dx^3}, \quad y''', \quad D_x^3[f(x)]$$

For n > 4, the nth derivative is written

$$f^{(n)}(x), \ y^{(n)}(x), \ \frac{d^ny}{dx^n}, \ or \ \frac{d^n}{dx^n}[f(x)], .$$

(See the caution in the middle of page 275 of the text.)

Ex 1: (a) If $f(x) = 3x^3 - 4x^2 + 5x - 8$; find the 1^{st}, 2^{nd}, 3^{rd}, and 4^{th} derivatives. (b) Find $f''(-2)$ and $f'''(1000)$

$a)$

$$f'(x) = 9x^2 - 8x + 5$$
$$f''(x) = 18x - 8$$
$$f'''(x) = 18$$
$$f^{(4)} = 0$$

$b)$

$$f''(-2) = 18(-2) - 8$$
$$= -36 - 8$$
$$= -44$$
$$f'''(1000) = 18$$

Ex 2: Find the first and second derivatives of $y = 2(x^3 - 2)^4$.

Ex 3: Find the first three derivatives of $g(x) = 2x^5 - \frac{3}{x^2}$

Ex 4: Find the first and second derivatives of each.

$a)\quad y = \frac{x+4}{x-3}$

$b)\quad f(n) = \frac{-5}{(n+3)^2}$

<u>Ex 5</u>: Find the second derivative two different ways; (1) using the product rule, and (2) using the basic rules after finding the product of the factors.

$$g(x) = 2x^3(x^2 - 3x + 2)$$

Ex 6: Find the second derivative of each.

$a)$ $f(x) = 2x(e^x)$

$b)$ $g(x) = \dfrac{\ln x}{2x}$

Remember a derivative is a rate of change. In the real world, the rate of change of a distance (position of a vehicle along a straight line) is **velocity**. $(v(t) = s'(t))$ The instantaneous rate of change of velocity is called **acceleration**. $a(t) = \dfrac{d}{dt}v(t) = v'(t) = s''(t)$

This is paragraph from page 276 of the textbook.

(1) If the velocity is positive and the acceleration is positive, the velocity is increasing, so the vehicle is speeding up.
(2) If the velocity is positive and the acceleration is negative, the vehicle is slowing down.
(3) A negative velocity and a positive acceleration mean the vehicle is backing up and slowing down.
(4) If both the velocity and acceleration are negative, the vehicle is speeding up in the negative direction.

Ex 7:
Suppose a truck is moving in a straight line, with its position from a starting point (in feet) at time *t* (in seconds) given by
$s(t) = t^3 - 3t^2 - 5t + 8$.

a) Find the velocity function and the specific velocity at time 8 seconds.

b) Find the acceleration function and the specific acceleration at time 8 seconds.

Concavity of a Graph: As easy description of concavity is the following. If the graph is 'concave upward', it 'holds' water. If the graph is 'concave downward', it 'spills' water. Figure 28 found on page 277 of the text (and shown below), illustrates this idea.

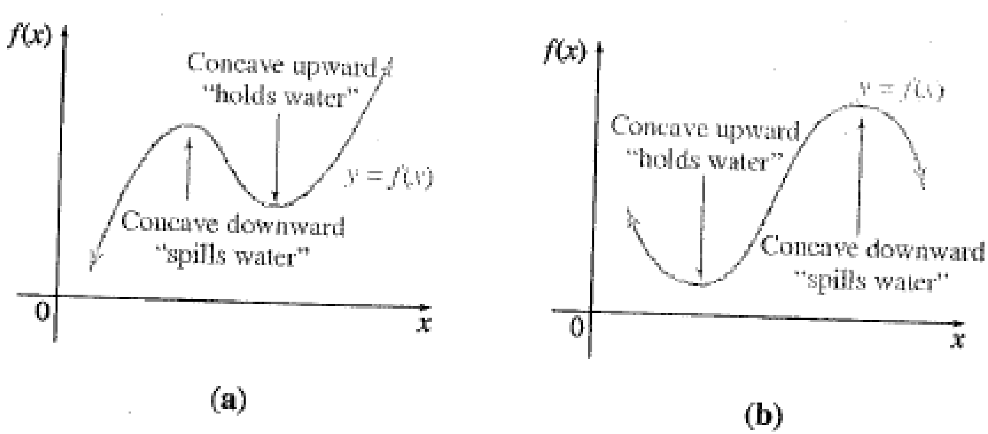


FIGURE 28

In figure 36 below, the graph is concave upward in the interval from x = -∞ to x = 1, illustrated by the interval $(-\infty, 1)$. The graph is also concave upward from x = 3 to infinity, illustrated by the interval $(3, \infty)$. The graph is concave downward in the interval (1, 3). Notice: The intervals of concavity are given from one x value to another x value.

$f(x) = x^4 - 8x^3 + 18x^2$

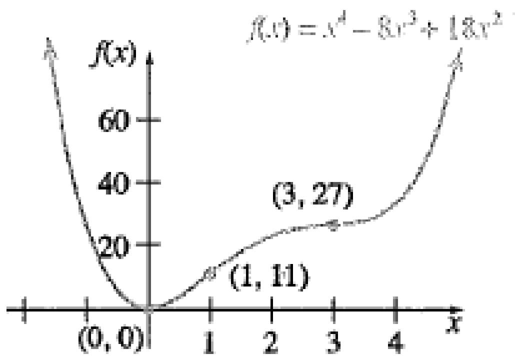


FIGURE 36

Figure 36 is also found on page 280.

The point where a graph (function) changes from concave one way to concave the other way is called an **inflection point or point of inflection.** The graph in figure 36 above (and on page 280 of the text) has two inflection points, (1, 11) and (3, 27).

One other point about concavity

Remember that the first derivative is the slope of a tangent line to a curve. Suppose the tangent lines to a curve are drawn as *x* gets larger. What happens to the slopes of the tangent lines?

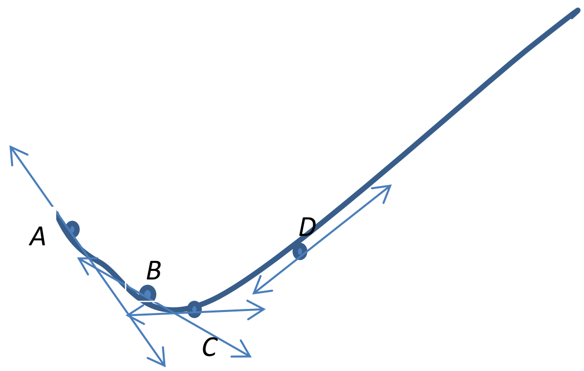


FIGURE 1

The slope of the tangent line at point *A* is negative. At point *B* (which is greater than *A*), the slope is negative, but a larger numeric value. At point *C*, the slope is almost 0 (larger than the slopes at points *A* and *B*). At point *D* the slope of the tangent line is positive. Conclusion: As *x* gets larger (*A* to *B* to *C* to *D*, left to right), the slopes of the tangent lines are getting larger. The change in the derivative is increasing.

The curve in figure 1 is **concave upward. Notice the curve lies <u>above</u> its tangent lines.**

Suppose the tangent lines to the curve below are drawn as *x* gets larger.

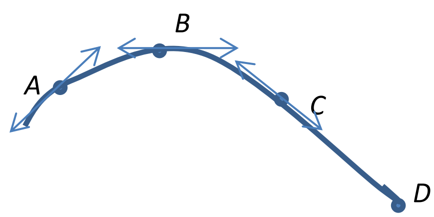


FIGURE 2

The slope of the tangent line at point *A* is a positive number. At point *B* is slope is still positive, but much smaller (almost zero). By point *C*, the slope is negative and at point *D*, the slope of the tangent line is an even smaller negative number. Conclusion: As *x* gets larger (*A* to *B* to *C* to *D*, left to right), the slopes of the tangent lines are getting smaller. The change in the derivative is decreasing.

The curve shown in figure 2 is **concave downward**. **Notice the curve lies <u>below</u> its tangent lines.**

<u>Test for Concavity</u>.

If *f* is a function with derivatives f' and f'' existing at all points in an interval (a, b). Then *f* if concave upward on (a, b) if $f'' > 0$ for all x in the interval and concave downward on (a, b) if $f'' < 0$ for all x in the interval.

<u>Ex 8</u>: Use the test for concavity and a **2nd derivative sign chart** to find where the function below is concave upward or concave downward.

$g(x) = x^4 - 8x^3 + 18x^2$

$g'(x) = 4x^3 - 24x^2 + 36x$

$g''(x) = 12x^2 - 48x + 36$

$12x^2 - 48x + 36 = 0$ Note: These values are not critical values.

$12(x^2 - 4x + 3) = 0$ They are simply values where the 2nd derivative is zero.

$12(x-3)(x-1) = 0$

$x - 3 = 0 \quad x - 1 = 0$

$x = 3 \quad x = 1$

	$(-\infty, 1)$	$(1, 3)$	$(3, \infty)$
12	$+$	$+$	$+$
$x-3$			
$x-1$			
Result			

Answer: Concave upward_____________

Concave downward___________

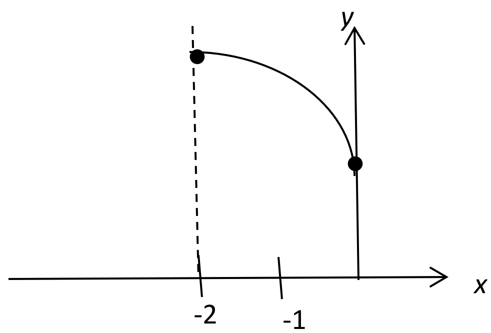
List any inflection points:_________________________

Ex 9: $f(x) = -2x^3 + 3x^2 + 72x$

a) Find critical values.
b) Find intervals of increasing or decreasing.
c) Find any relative extrema.
d) Find intervals of concavity.
e) Find any inflection points.

Example 10: Look at each graph of a function f. Determine the signs of $f'(x)$ and $f''(x)$ for the interval of each graph shown.

a)



b)