1

Introduction

In this chapter we give perspective to your study of differential equations in several different ways. First, we use two problems to illustrate some of the basic ideas that we will return to, and elaborate upon, frequently throughout the remainder of the book. Later, to provide organizational structure for the book, we indicate several ways of classifying differential equations. Finally, we outline some of the major trends in the historical development of the subject and mention a few of the outstanding mathematicians who have contributed to it. The study of differential equations has attracted the attention of many of the world's greatest mathematicians during the past three centuries. Nevertheless, it remains a dynamic field of inquiry today, with many interesting open questions.

1.1 Some Basic Mathematical Models; Direction Fields

Before embarking on a serious study of differential equations (for example, by reading this book or major portions of it), you should have some idea of the possible benefits to be gained by doing so. For some students the intrinsic interest of the subject itself is enough motivation, but for most it is the likelihood of important applications to other fields that makes the undertaking worthwhile.

Many of the principles, or laws, underlying the behavior of the natural world are statements or relations involving rates at which things happen. When expressed in mathematical terms, the relations are equations and the rates are derivatives. Equations containing derivatives are **differential equations**. Therefore, to understand and to investigate problems involving the motion of fluids, the flow of current in electric circuits, the dissipation of heat in solid objects, the propagation and detection of seismic waves, or the increase or decrease of populations, among many others, it is necessary to know something about differential equations.

A differential equation that describes some physical process is often called a **mathematical model** of the process, and many such models are discussed throughout this book. In this section we begin with two models leading to equations that are easy to solve. It is noteworthy that even the simplest differential equations provide useful models of important physical processes.

EXAMPLE 1

A Falling Object

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

We begin by introducing letters to represent various quantities that may be of interest in this problem. The motion takes place during a certain time interval, so let us use t to denote time. Also, let us use v to represent the velocity of the falling object. The velocity will presumably change with time, so we think of v as a function of t; in other words, t is the independent variable and v is the dependent variable. The choice of units of measurement is somewhat arbitrary, and there is nothing in the statement of the problem to suggest appropriate units, so we are free to make any choice that seems reasonable. To be specific, let us measure time t in seconds and velocity v in meters/second. Further, we will assume that v is positive in the downward direction—that is, when the object is falling.

The physical law that governs the motion of objects is Newton's second law, which states that the mass of the object times its acceleration is equal to the net force on the object. In mathematical terms this law is expressed by the equation

$$F = ma, \tag{1}$$

where m is the mass of the object, a is its acceleration, and F is the net force exerted on the object. To keep our units consistent, we will measure m in kilograms, a in meters/second^2, and F in newtons. Of course, a is related to v by $a = dv/dt$, so we can rewrite Eq. (1) in the form

$$F = m(dv/dt). \tag{2}$$

Next, consider the forces that act on the object as it falls. Gravity exerts a force equal to the weight of the object, or mg, where g is the acceleration due to gravity. In the units we have chosen, g has been determined experimentally to be approximately equal to 9.8 m/s^2 near the earth's surface. There is also a force due to air resistance, or drag, that is more difficult to model. This is not the place for an extended discussion of the drag force; suffice it to say that it is often assumed that the drag is proportional to the velocity, and we will make that assumption here. Thus the drag force has the magnitude γv, where γ is a constant called the drag coefficient. The numerical value of the drag coefficient varies widely from one object to another; smooth streamlined objects have much smaller drag coefficients than rough blunt ones. The physical units for γ are mass/time, or kg/s for this problem; if these units seem peculiar, remember that γv must have the units of force, namely, kg·m/s^2.

In writing an expression for the net force F, we need to remember that gravity always acts in the downward (positive) direction, whereas, for a falling object, drag acts in the upward (negative) direction, as shown in Figure 1.1.1. Thus

$$F = mg - \gamma v \tag{3}$$

and Eq. (2) then becomes

$$m\frac{dv}{dt} = mg - \gamma v. \tag{4}$$

Equation (4) is a mathematical model of an object falling in the atmosphere near sea level. Note that the model contains the three constants m, g, and γ. The constants m and γ depend very much on the particular object that is falling, and they are usually different for different objects. It is common to refer to them as parameters, since they may take on a range of values during the course of an experiment. On the other hand, g is a physical constant, whose value is the same for all objects.

FIGURE 1.1.1 Free-body diagram of the forces on a falling object.

To solve Eq. (4), we need to find a function $v = v(t)$ that satisfies the equation. It is not hard to do this, and we will show you how in the next section. For the present, however, let us see what we can learn about solutions without actually finding any of them. Our task is simplified slightly if we assign numerical values to m and γ, but the procedure is the same regardless of which values we choose. So, let us suppose that $m = 10$ kg and $\gamma = 2$ kg/s. Then Eq. (4) can be rewritten as

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{5}$$

EXAMPLE 2

A Falling Object (continued)

Investigate the behavior of solutions of Eq. (5) without solving the differential equation.

First let us consider what information can be obtained directly from the differential equation itself. Suppose that the velocity v has a certain given value. Then, by evaluating the right side of Eq. (5), we can find the corresponding value of dv/dt. For instance, if $v = 40$, then $dv/dt = 1.8$. This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $v = 40$. We can display this information graphically in the tv-plane by drawing short line segments with slope 1.8 at several points on the line $v = 40$. Similarly, if $v = 50$, then $dv/dt = -0.2$, so we draw line segments with slope -0.2 at several points on the line $v = 50$. We obtain Figure 1.1.2 by proceeding in the same way with other values of v. Figure 1.1.2 is an example of what is called a **direction field** or sometimes a **slope field**.

Remember that a solution of Eq. (5) is a function $v = v(t)$ whose graph is a curve in the tv-plane. The importance of Figure 1.1.2 is that each line segment is a tangent line to one of these solution curves. Thus, even though we have not found any solutions, and no graphs of solutions appear in the figure, we can nonetheless draw some qualitative conclusions about the behavior of solutions. For instance, if v is less than a certain critical value, then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value, then the line segments have negative slopes, and the falling object slows down as it falls. What is this critical value of v that separates objects whose speed is increasing from those whose speed is decreasing? Referring again to Eq. (5), we ask what value of v will cause dv/dt to be zero. The answer is $v = (5)(9.8) = 49$ m/s.

In fact, the constant function $v(t) = 49$ is a solution of Eq. (5). To verify this statement, substitute $v(t) = 49$ into Eq. (5) and observe that each side of the equation is zero. Because it does not change with time, the solution $v(t) = 49$ is called an **equilibrium solution**. It is the solution that corresponds to a perfect balance between gravity and drag. In Figure 1.1.3

we show the equilibrium solution $v(t) = 49$ superimposed on the direction field. From this figure we can draw another conclusion, namely, that all other solutions seem to be converging to the equilibrium solution as t increases. Thus, in this context, the equilibrium solution is often called the **terminal velocity**.

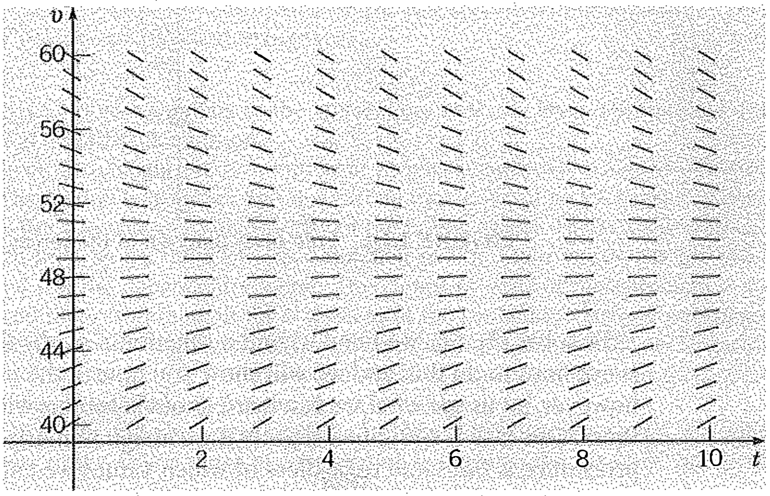

FIGURE 1.1.2 A direction field for Eq. (5): $dv/dt = 9.8 - (v/5)$.

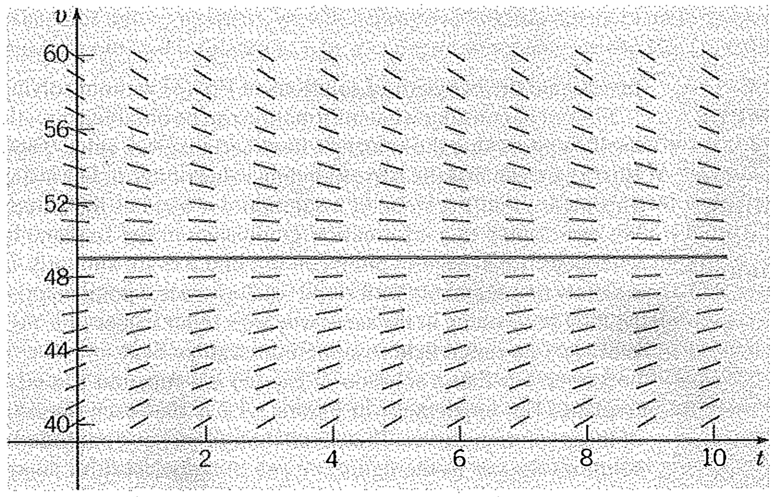

FIGURE 1.1.3 Direction field and equilibrium solution for Eq. (5): $dv/dt = 9.8 - (v/5)$.

The approach illustrated in Example 2 can be applied equally well to the more general Eq. (4), where the parameters m and γ are unspecified positive numbers. The results are essentially identical to those of Example 2. The equilibrium solution of Eq. (4) is $v(t) = mg/\gamma$. Solutions below the equilibrium solution increase with time, those above it decrease with time, and all other solutions approach the equilibrium solution as t becomes large.

Direction Fields. Direction fields are valuable tools in studying the solutions of differential equations of the form

$$\frac{dy}{dt} = f(t, y), \tag{6}$$

where f is a given function of the two variables t and y, sometimes referred to as the **rate function**. A direction field for equations of the form (6) can be constructed by evaluating f at each point of a rectangular grid. At each point of the grid, a short line segment is drawn whose slope is the value of f at that point. Thus each line segment is tangent to the graph of the solution passing through that point. A direction field drawn on a fairly fine grid gives a good picture of the overall behavior of solutions of a differential equation. Usually a grid consisting of a few hundred points is sufficient. The construction of a direction field is often a useful first step in the investigation of a differential equation.

Two observations are worth particular mention. First, in constructing a direction field, we do not have to solve Eq. (6); we just have to evaluate the given function $f(t, y)$ many times. Thus direction fields can be readily constructed even for equations that may be quite difficult to solve. Second, repeated evaluation of a given function is a task for which a computer is well suited, and you should usually use a computer to draw a direction field. All the direction fields shown in this book, such as the one in Figure 1.1.2, were computer-generated.

Field Mice and Owls. Now let us look at another, quite different example. Consider a population of field mice who inhabit a certain rural area. In the absence of predators we assume that the mouse population increases at a rate proportional to the current population. This assumption is not a well-established physical law (as Newton's law of motion is in Example 1), but it is a common initial hypothesis[1] in a study of population growth. If we denote time by t and the mouse population by $p(t)$, then the assumption about population growth can be expressed by the equation

$$\frac{dp}{dt} = rp, \tag{7}$$

where the proportionality factor r is called the **rate constant** or **growth rate**. To be specific, suppose that time is measured in months and that the rate constant r has the value 0.5/month. Then each term in Eq. (7) has the units of mice/month.

Now let us add to the problem by supposing that several owls live in the same neighborhood and that they kill 15 field mice per day. To incorporate this information into the model, we must add another term to the differential equation (7), so that it becomes

$$\frac{dp}{dt} = 0.5p - 450. \tag{8}$$

Observe that the predation term is -450 rather than -15 because time is measured in months, so the monthly predation rate is needed.

[1] A better model of population growth is discussed in Section 2.5.

EXAMPLE 3

Investigate the solutions of Eq. (8) graphically.

A direction field for Eq. (8) is shown in Figure 1.1.4. For sufficiently large values of p it can be seen from the figure, or directly from Eq. (8) itself, that dp/dt is positive, so that solutions increase. On the other hand, if p is small, then dp/dt is negative and solutions decrease. Again, the critical value of p that separates solutions that increase from those that decrease is the value of p for which dp/dt is zero. By setting dp/dt equal to zero in Eq. (8) and then solving for p, we find the equilibrium solution $p(t) = 900$, for which the growth term and the predation term in Eq. (8) are exactly balanced. The equilibrium solution is also shown in Figure 1.1.4.

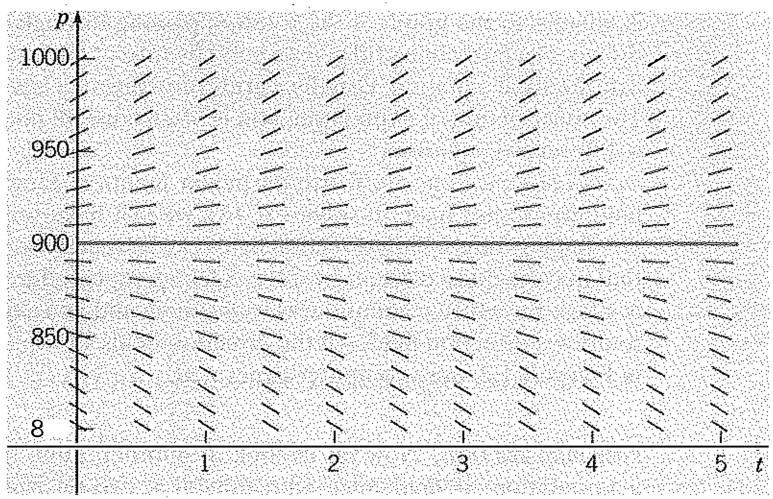

FIGURE 1.1.4 Direction field and equilibrium solution for Eq. (8): $dp/dt = 0.5p - 450$.

Comparing Examples 2 and 3, we note that in both cases the equilibrium solution separates increasing from decreasing solutions. In Example 2 other solutions converge to, or are attracted by, the equilibrium solution, so that after the object falls far enough, an observer will see it moving at very nearly the equilibrium velocity. On the other hand, in Example 3 other solutions diverge from, or are repelled by, the equilibrium solution. Solutions behave very differently depending on whether they start above or below the equilibrium solution. As time passes, an observer might see populations either much larger or much smaller than the equilibrium population, but the equilibrium solution itself will not, in practice, be observed. In both problems, however, the equilibrium solution is very important in understanding how solutions of the given differential equation behave.

A more general version of Eq. (8) is

$$\frac{dp}{dt} = rp - k, \tag{9}$$

where the growth rate r and the predation rate k are unspecified. Solutions of this more general equation are very similar to those of Eq. (8). The equilibrium solution of Eq. (9) is $p(t) = k/r$. Solutions above the equilibrium solution increase, while those below it decrease.

You should keep in mind that both of the models discussed in this section have their limitations. The model (5) of the falling object is valid only as long as the object is falling freely, without encountering any obstacles. The population model (8) eventually predicts negative numbers of mice (if $p < 900$) or enormously large numbers (if $p > 900$). Both of these predictions are unrealistic, so this model becomes unacceptable after a fairly short time interval.

Constructing Mathematical Models. In applying differential equations to any of the numerous fields in which they are useful, it is necessary first to formulate the appropriate differential equation that describes, or models, the problem being investigated. In this section we have looked at two examples of this modeling process, one drawn from physics and the other from ecology. In constructing future mathematical models yourself, you should recognize that each problem is different, and that successful modeling cannot be reduced to the observance of a set of prescribed rules. Indeed, constructing a satisfactory model is sometimes the most difficult part of the problem. Nevertheless, it may be helpful to list some steps that are often part of the process:

1. Identify the independent and dependent variables and assign letters to represent them. Often the independent variable is time.
2. Choose the units of measurement for each variable. In a sense the choice of units is arbitrary, but some choices may be much more convenient than others. For example, we chose to measure time in seconds for the falling-object problem and in months for the population problem.
3. Articulate the basic principle that underlies or governs the problem you are investigating. This may be a widely recognized physical law, such as Newton's law of motion, or it may be a more speculative assumption that may be based on your own experience or observations. In any case, this step is likely not to be a purely mathematical one, but will require you to be familiar with the field in which the problem originates.
4. Express the principle or law in step 3 in terms of the variables you chose in step 1. This may be easier said than done. It may require the introduction of physical constants or parameters (such as the drag coefficient in Example 1) and the determination of appropriate values for them. Or it may involve the use of auxiliary or intermediate variables that must then be related to the primary variables.
5. Make sure that all terms in your equation have the same physical units. If this is not the case, then your equation is wrong and you should seek to repair it. If the units agree, then your equation at least is dimensionally consistent, although it may have other shortcomings that this test does not reveal.
6. In the problems considered here, the result of step 4 is a single differential equation, which constitutes the desired mathematical model. Keep in mind, though, that in more complex problems the resulting mathematical model may be much more complicated, perhaps involving a system of several differential equations, for example.

PROBLEMS

In each of Problems 1 through 6, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe the dependency.

1. $y' = 3 - 2y$
2. $y' = 2y - 3$
3. $y' = 3 + 2y$
4. $y' = -1 - 2y$
5. $y' = 1 + 2y$
6. $y' = y + 2$

In each of Problems 7 through 10, write down a differential equation of the form $dy/dt = ay + b$ whose solutions have the required behavior as $t \to \infty$.

7. All solutions approach $y = 3$.
8. All solutions approach $y = 2/3$.
9. All other solutions diverge from $y = 2$.
10. All other solutions diverge from $y = 1/3$.

In each of Problems 11 through 14, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that in these problems the equations are not of the form $y' = ay + b$, and the behavior of their solutions is somewhat more complicated than for the equations in the text.

11. $y' = y(4 - y)$
12. $y' = -y(5 - y)$
13. $y' = y^2$
14. $y' = y(y - 2)^2$

Consider the following list of differential equations, some of which produced the direction fields shown in Figures 1.1.5 through 1.1.10. In each of Problems 15 through 20 identify the differential equation that corresponds to the given direction field.

(a) $y' = 2y - 1$
(b) $y' = 2 + y$
(c) $y' = y - 2$
(d) $y' = y(y + 3)$
(e) $y' = y(y - 3)$
(f) $y' = 1 + 2y$
(g) $y' = -2 - y$
(h) $y' = y(3 - y)$
(i) $y' = 1 - 2y$
(j) $y' = 2 - y$

15. The direction field of Figure 1.1.5.
16. The direction field of Figure 1.1.6.

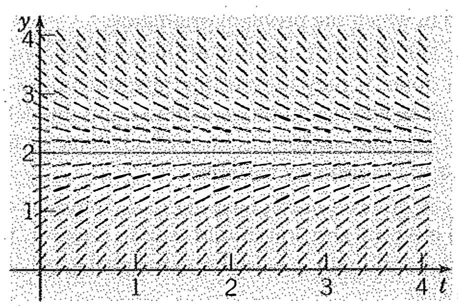

FIGURE 1.1.5 Problem 15.

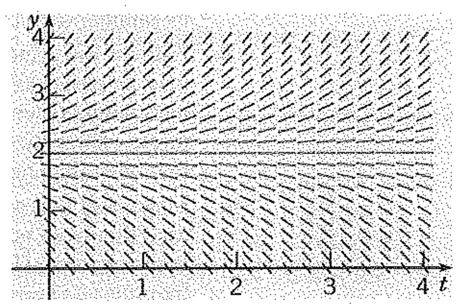

FIGURE 1.1.6 Problem 16.

17. The direction field of Figure 1.1.7.
18. The direction field of Figure 1.1.8.

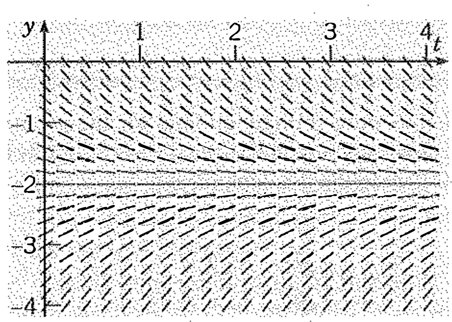

FIGURE 1.1.7 Problem 17.

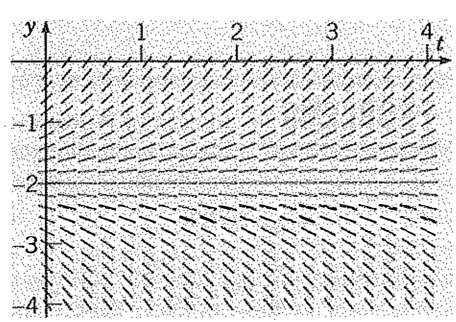

FIGURE 1.1.8 Problem 18.

19. The direction field of Figure 1.1.9.
20. The direction field of Figure 1.1.10.

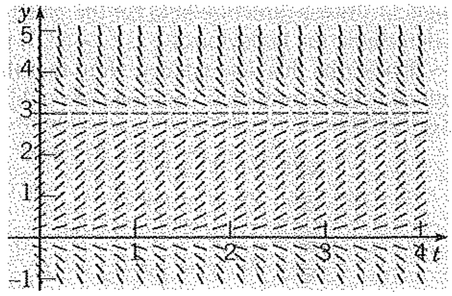

FIGURE 1.1.9 Problem 19.

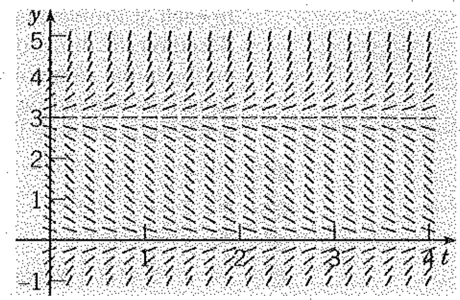

FIGURE 1.1.10 Problem 20.

21. A pond initially contains 1,000,000 gal of water and an unknown amount of an undesirable chemical. Water containing 0.01 g of this chemical per gallon flows into the pond at a rate of 300 gal/h. The mixture flows out at the same rate, so the amount of water in the pond remains constant. Assume that the chemical is uniformly distributed throughout the pond.
    (a) Write a differential equation for the amount of chemical in the pond at any time.
    (b) How much of the chemical will be in the pond after a very long time? Does this limiting amount depend on the amount that was present initially?
22. A spherical raindrop evaporates at a rate proportional to its surface area. Write a differential equation for the volume of the raindrop as a function of time.
23. Newton's law of cooling states that the temperature of an object changes at a rate proportional to the difference between the temperature of the object itself and the temperature of its surroundings (the ambient air temperature in most cases). Suppose that the ambient temperature is 70°F and that the rate constant is 0.05 $(\text{min})^{-1}$. Write a differential equation for the temperature of the object at any time. Note that the differential equation is the same whether the temperature of the object is above or below the ambient temperature.
24. A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm^3 of the drug enters the patient's bloodstream at a rate of 100 cm^3/h. The drug is absorbed by body tissues or otherwise leaves the bloodstream at a rate proportional to the amount present, with a rate constant of 0.4 $(\text{h})^{-1}$.
    (a) Assuming that the drug is always uniformly distributed throughout the bloodstream, write a differential equation for the amount of the drug that is present in the bloodstream at any time.
    (b) How much of the drug is present in the bloodstream after a long time?
25. For small, slowly falling objects, the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects, it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]
    (a) Write a differential equation for the velocity of a falling object of mass m if the magnitude of the drag force is proportional to the square of the velocity and its direction is opposite to that of the velocity.

[2]See Lyle N. Long and Howard Weiss, "The Velocity Dependence of Aerodynamic Drag: A Primer for Mathematicians," *American Mathematical Monthly 106* (1999), 2, pp. 127–135.

(b) Determine the limiting velocity after a long time.

(c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/s.

(d) Using the data in part (c), draw a direction field and compare it with Figure 1.1.3.

In each of Problems 26 through 33, draw a direction field for the given differential equation. Based on the direction field, determine the behavior of y as $t \to \infty$. If this behavior depends on the initial value of y at $t = 0$, describe this dependency. Note that the right sides of these equations depend on t as well as y; therefore, their solutions can exhibit more complicated behavior than those in the text.

26. $y' = -2 + t - y$
27. $y' = te^{-2t} - 2y$
28. $y' = e^{-t} + y$
29. $y' = t + 2y$
30. $y' = 3\sin t + 1 + y$
31. $y' = 2t - 1 - y^2$
32. $y' = -(2t + y)/2y$
33. $y' = \frac{1}{6}y^3 - y - \frac{1}{3}t^2$

1.2 Solutions of Some Differential Equations

In the preceding section we derived the differential equations

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

and

$$\frac{dp}{dt} = rp - k. \tag{2}$$

Equation (1) models a falling object, and Eq. (2) models a population of field mice preyed on by owls. Both of these equations are of the general form

$$\frac{dy}{dt} = ay - b, \tag{3}$$

where a and b are given constants. We were able to draw some important qualitative conclusions about the behavior of solutions of Eqs. (1) and (2) by considering the associated direction fields. To answer questions of a quantitative nature, however, we need to find the solutions themselves, and we now investigate how to do that.

EXAMPLE 1

Field Mice and Owls (continued)

Consider the equation

$$\frac{dp}{dt} = 0.5p - 450, \tag{4}$$

which describes the interaction of certain populations of field mice and owls [see Eq. (8) of Section 1.1]. Find solutions of this equation.

To solve Eq. (4), we need to find functions $p(t)$ that, when substituted into the equation, reduce it to an obvious identity. Here is one way to proceed. First, rewrite Eq. (4) in the form

$$\frac{dp}{dt} = \frac{p - 900}{2}, \tag{5}$$

or, if $p \neq 900$,

$$\frac{dp/dt}{p - 900} = \frac{1}{2}. \tag{6}$$

By the chain rule the left side of Eq. (6) is the derivative of $\ln|p-900|$ with respect to t, so we have

$$\frac{d}{dt}\ln|p-900| = \frac{1}{2}. \tag{7}$$

Then, by integrating both sides of Eq. (7), we obtain

$$\ln|p-900| = \frac{t}{2} + C, \tag{8}$$

where C is an arbitrary constant of integration. Therefore, by taking the exponential of both sides of Eq. (8), we find that

$$|p-900| = e^{(t/2)+C} = e^C e^{t/2}, \tag{9}$$

or

$$p - 900 = \pm e^C e^{t/2}, \tag{10}$$

and finally

$$p = 900 + ce^{t/2}, \tag{11}$$

where $c = \pm e^C$ is also an arbitrary (nonzero) constant. Note that the constant function $p = 900$ is also a solution of Eq. (5) and that it is contained in the expression (11) if we allow c to take the value zero. Graphs of Eq. (11) for several values of c are shown in Figure 1.2.1.

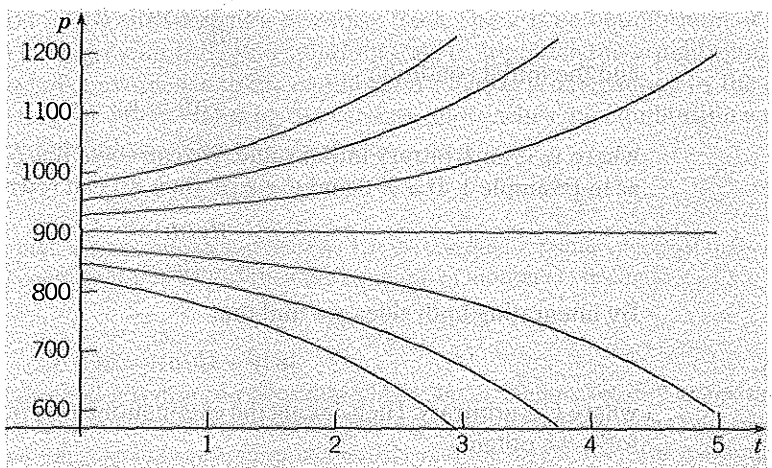

FIGURE 1.2.1 Graphs of $p = 900 + ce^{t/2}$ for several values of c. These are solutions of $dp/dt = 0.5p - 450$.

Note that they have the character inferred from the direction field in Figure 1.1.4. For instance, solutions lying on either side of the equilibrium solution $p = 900$ tend to diverge from that solution.

In Example 1 we found infinitely many solutions of the differential equation (4), corresponding to the infinitely many values that the arbitrary constant c in Eq. (11) might have. This is typical of what happens when you solve a differential equation. The solution process involves an integration, which brings with it an arbitrary constant, whose possible values generate an infinite family of solutions.

Frequently, we want to focus our attention on a single member of the infinite family of solutions by specifying the value of the arbitrary constant. Most often, we do this indirectly by specifying instead a point that must lie on the graph of the solution. For example, to determine the constant c in Eq. (11), we could require that the population have a given value at a certain time, such as the value 850 at time $t = 0$. In other words, the graph of the solution must pass through the point (0, 850). Symbolically, we can express this condition as

$$p(0) = 850. \tag{12}$$

Then, substituting $t = 0$ and $p = 850$ into Eq. (11), we obtain

$$850 = 900 + c.$$

Hence $c = -50$, and by inserting this value into Eq. (11), we obtain the desired solution, namely,

$$p = 900 - 50e^{t/2}. \tag{13}$$

The additional condition (12) that we used to determine c is an example of an **initial condition.** The differential equation (4) together with the initial condition (12) form an **initial value problem.**

Now consider the more general problem consisting of the differential equation (3)

$$\frac{dy}{dt} = ay - b$$

and the initial condition

$$y(0) = y_0, \tag{14}$$

where y_0 is an arbitrary initial value. We can solve this problem by the same method as in Example 1. If $a \neq 0$ and $y \neq b/a$, then we can rewrite Eq. (3) as

$$\frac{dy/dt}{y - (b/a)} = a. \tag{15}$$

By integrating both sides, we find that

$$\ln|y - (b/a)| = at + C, \tag{16}$$

where C is arbitrary. Then, taking the exponential of both sides of Eq. (16) and solving for y, we obtain

$$y = (b/a) + ce^{at}, \tag{17}$$

where $c = \pm e^C$ is also arbitrary. Observe that $c = 0$ corresponds to the equilibrium solution $y = b/a$. Finally, the initial condition (14) requires that $c = y_0 - (b/a)$, so the solution of the initial value problem (3), (14) is

$$y = (b/a) + [y_0 - (b/a)]e^{at}. \tag{18}$$

For $a \neq 0$ the expression (17) contains all possible solutions of Eq. (3) and is called the **general solution**.[3] The geometrical representation of the general solution (17) is an infinite family of curves called **integral curves.** Each integral curve is associated with a particular value of c and is the graph of the solution corresponding to that

[3] If $a = 0$, then the solution of Eq. (3) is not given by Eq. (17). We leave it to you to find the general solution in this case.

value of c. Satisfying an initial condition amounts to identifying the integral curve that passes through the given initial point.

To relate the solution (18) to Eq. (2), which models the field mouse population, we need only replace a by the growth rate r and replace b by the predation rate k. Then the solution (18) becomes

$$p = (k/r) + [p_0 - (k/r)]e^{rt}, \tag{19}$$

where p_0 is the initial population of field mice. The solution (19) confirms the conclusions reached on the basis of the direction field and Example 1. If $p_0 = k/r$, then from Eq. (19) it follows that $p = k/r$ for all t; this is the constant, or equilibrium, solution. If $p_0 \neq k/r$, then the behavior of the solution depends on the sign of the coefficient $p_0 - (k/r)$ of the exponential term in Eq. (19). If $p_0 > k/r$, then p grows exponentially with time t; if $p_0 < k/r$, then p decreases and eventually becomes zero, corresponding to extinction of the field mouse population. Negative values of p, while possible for the expression (19), make no sense in the context of this particular problem.

To put the falling-object equation (1) in the form (3), we must identify a with $-\gamma/m$ and b with $-g$. Making these substitutions in the solution (18), we obtain

$$v = (mg/\gamma) + [v_0 - (mg/\gamma)]e^{-\gamma t/m}, \tag{20}$$

where v_0 is the initial velocity. Again, this solution confirms the conclusions reached in Section 1.1 on the basis of a direction field. There is an equilibrium, or constant, solution $v = mg/\gamma$, and all other solutions tend to approach this equilibrium solution. The speed of convergence to the equilibrium solution is determined by the exponent $-\gamma/m$. Thus, for a given mass m, the velocity approaches the equilibrium value more rapidly as the drag coefficient γ increases.

EXAMPLE 2

A Falling Object (continued)

Suppose that, as in Example 2 of Section 1.1, we consider a falling object of mass $m = 10$ kg and drag coefficient $\gamma = 2$ kg/s. Then the equation of motion (1) becomes

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}. \tag{21}$$

Suppose this object is dropped from a height of 300 m. Find its velocity at any time t. How long will it take to fall to the ground, and how fast will it be moving at the time of impact?

The first step is to state an appropriate initial condition for Eq. (21). The word "dropped" in the statement of the problem suggests that the initial velocity is zero, so we will use the initial condition

$$v(0) = 0. \tag{22}$$

The solution of Eq. (21) can be found by substituting the values of the coefficients into the solution (20), but we will proceed instead to solve Eq. (21) directly. First, rewrite the equation as

$$\frac{dv/dt}{v - 49} = -\frac{1}{5}. \tag{23}$$

By integrating both sides, we obtain

$$\ln |v - 49| = -\frac{t}{5} + C, \tag{24}$$

and then the general solution of Eq. (21) is

$$v = 49 + ce^{-t/5}, \tag{25}$$

where c is arbitrary. To determine c, we substitute $t = 0$ and $v = 0$ from the initial condition (22) into Eq. (25), with the result that $c = -49$. Then the solution of the initial value problem (21), (22) is

$$v = 49(1 - e^{-t/5}). \tag{26}$$

Equation (26) gives the velocity of the falling object at any positive time (before it hits the ground, of course).

Graphs of the solution (25) for several values of c are shown in Figure 1.2.2, with the solution (26) shown by the black curve. It is evident that all solutions tend to approach the equilibrium solution $v = 49$. This confirms the conclusions we reached in Section 1.1 on the basis of the direction fields in Figures 1.1.2 and 1.1.3.

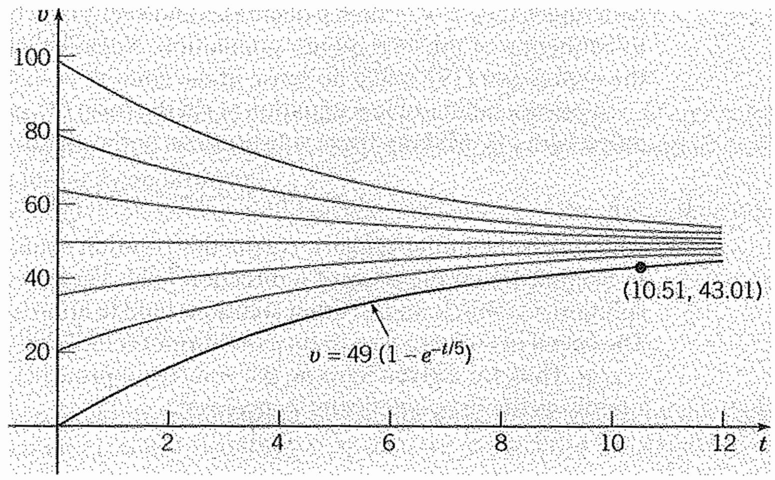

FIGURE 1.2.2 Graphs of the solution (25), $v = 49 + ce^{-t/5}$, for several values of c. The black curve corresponds to the initial condition $v(0) = 0$.

To find the velocity of the object when it hits the ground, we need to know the time at which impact occurs. In other words, we need to determine how long it takes the object to fall 300 m. To do this, we note that the distance x the object has fallen is related to its velocity v by the equation $v = dx/dt$, or

$$\frac{dx}{dt} = 49(1 - e^{-t/5}). \tag{27}$$

Consequently, by integrating both sides of Eq. (27), we have

$$x = 49t + 245e^{-t/5} + c, \tag{28}$$

where c is an arbitrary constant of integration. The object starts to fall when $t = 0$, so we know that $x = 0$ when $t = 0$. From Eq. (28) it follows that $c = -245$, so the distance the object has fallen at time t is given by

$$x = 49t + 245e^{-t/5} - 245. \tag{29}$$

Let T be the time at which the object hits the ground; then $x = 300$ when $t = T$. By substituting these values in Eq. (29), we obtain the equation

$$49T + 245e^{-T/5} - 545 = 0. \tag{30}$$

The value of T satisfying Eq. (30) can be approximated by a numerical process[4] using a scientific calculator or computer, with the result that $T \cong 10.51$ s. At this time, the corresponding velocity v_T is found from Eq. (26) to be $v_T \cong 43.01$ m/s. The point (10.51, 43.01) is also shown in Figure 1.2.2.

Further Remarks on Mathematical Modeling. Up to this point we have related our discussion of differential equations to mathematical models of a falling object and of a hypothetical relation between field mice and owls. The derivation of these models may have been plausible, and possibly even convincing, but you should remember that the ultimate test of any mathematical model is whether its predictions agree with observations or experimental results. We have no actual observations or experimental results to use for comparison purposes here, but there are several sources of possible discrepancies.

In the case of the falling object, the underlying physical principle (Newton's law of motion) is well established and widely applicable. However, the assumption that the drag force is proportional to the velocity is less certain. Even if this assumption is correct, the determination of the drag coefficient γ by direct measurement presents difficulties. Indeed, sometimes one finds the drag coefficient indirectly—for example, by measuring the time of fall from a given height and then calculating the value of γ that predicts this observed time.

The model of the field mouse population is subject to various uncertainties. The determination of the growth rate r and the predation rate k depends on observations of actual populations, which may be subject to considerable variation. The assumption that r and k are constants may also be questionable. For example, a constant predation rate becomes harder to sustain as the field mouse population becomes smaller. Further, the model predicts that a population above the equilibrium value will grow exponentially larger and larger. This seems at variance with the behavior of actual populations; see the further discussion of population dynamics in Section 2.5.

If the differences between actual observations and a mathematical model's predictions are too great, then you need to consider refining the model, making more careful observations, or perhaps both. There is almost always a tradeoff between accuracy and simplicity. Both are desirable, but a gain in one usually involves a loss in the other. However, even if a mathematical model is incomplete or somewhat inaccurate, it may nevertheless be useful in explaining qualitative features of the problem under investigation. It may also give satisfactory results under some circumstances but not others. Thus you should always use good judgment and common sense in constructing mathematical models and in using their predictions.

PROBLEMS

1. Solve each of the following initial value problems and plot the solutions for several values of y_0. Then describe in a few words how the solutions resemble, and differ from, each other.

   (a) $dy/dt = -y + 5, \qquad y(0) = y_0$

[4] A computer algebra system provides this capability; many calculators also have built-in routines for solving such equations.

(b) $dy/dt = -2y + 5, \quad y(0) = y_0$
(c) $dy/dt = -2y + 10, \quad y(0) = y_0$

2. Follow the instructions for Problem 1 for the following initial value problems:
(a) $dy/dt = y - 5, \quad y(0) = y_0$
(b) $dy/dt = 2y - 5, \quad y(0) = y_0$
(c) $dy/dt = 2y - 10, \quad y(0) = y_0$

3. Consider the differential equation

$$dy/dt = -ay + b,$$

where both a and b are positive numbers.
(a) Find the general solution of the differential equation.
(b) Sketch the solution for several different initial conditions.
(c) Describe how the solutions change under each of the following conditions:
i. a increases.
ii. b increases.
iii. Both a and b increase, but the ratio b/a remains the same.

4. Consider the differential equation $dy/dt = ay - b$.
(a) Find the equilibrium solution y_e.
(b) Let $Y(t) = y - y_e$; thus $Y(t)$ is the deviation from the equilibrium solution. Find the differential equation satisfied by $Y(t)$.

5. **Undetermined Coefficients.** Here is an alternative way to solve the equation

$$dy/dt = ay - b. \tag{i}$$

(a) Solve the simpler equation

$$dy/dt = ay. \tag{ii}$$

Call the solution $y_1(t)$.
(b) Observe that the only difference between Eqs. (i) and (ii) is the constant $-b$ in Eq. (i). Therefore, it may seem reasonable to assume that the solutions of these two equations also differ only by a constant. Test this assumption by trying to find a constant k such that $y = y_1(t) + k$ is a solution of Eq. (i).
(c) Compare your solution from part (b) with the solution given in the text in Eq. (17).
Note: This method can also be used in some cases in which the constant b is replaced by a function $g(t)$. It depends on whether you can guess the general form that the solution is likely to take. This method is described in detail in Section 3.5 in connection with second order equations.

6. Use the method of Problem 5 to solve the equation

$$dy/dt = -ay + b.$$

7. The field mouse population in Example 1 satisfies the differential equation

$$dp/dt = 0.5p - 450.$$

(a) Find the time at which the population becomes extinct if $p(0) = 850$.
(b) Find the time of extinction if $p(0) = p_0$, where $0 < p_0 < 900$.
(c) Find the initial population p_0 if the population is to become extinct in 1 year.

8. Consider a population p of field mice that grows at a rate proportional to the current population, so that $dp/dt = rp$.
   (a) Find the rate constant r if the population doubles in 30 days.
   (b) Find r if the population doubles in N days.

9. The falling object in Example 2 satisfies the initial value problem

$$dv/dt = 9.8 - (v/5), \qquad v(0) = 0.$$

   (a) Find the time that must elapse for the object to reach 98% of its limiting velocity.
   (b) How far does the object fall in the time found in part (a)?

10. Modify Example 2 so that the falling object experiences no air resistance.
    (a) Write down the modified initial value problem.
    (b) Determine how long it takes the object to reach the ground.
    (c) Determine its velocity at the time of impact.

11. Consider the falling object of mass 10 kg in Example 2, but assume now that the drag force is proportional to the square of the velocity.
    (a) If the limiting velocity is 49 m/s (the same as in Example 2), show that the equation of motion can be written as

$$dv/dt = [(49)^2 - v^2]/245.$$

    Also see Problem 25 of Section 1.1.
    (b) If $v(0) = 0$, find an expression for $v(t)$ at any time.
    (c) Plot your solution from part (b) and the solution (26) from Example 2 on the same axes.
    (d) Based on your plots in part (c), compare the effect of a quadratic drag force with that of a linear drag force.
    (e) Find the distance $x(t)$ that the object falls in time t.
    (f) Find the time T it takes the object to fall 300 m.

12. A radioactive material, such as the isotope thorium-234, disintegrates at a rate proportional to the amount cmrently present. If $Q(t)$ is the amount present at time t, then $dQ/dt = -rQ$, where $r > 0$ is the decay rate.
    (a) If 100 mg of thorium-234 decays to 82.04 mg in 1 week, determine the decay rate r.
    (b) Find an expression for the amount of thorium-234 present at any time t.
    (c) Find the time required for the thorium-234 to decay to one-half its original amount.

13. The **half-life** of a radioactive material is the time required for an amount of this material to decay to one-half its original value. Show that for any radioactive material that decays according to the equation $Q' = -rQ$, the half-life τ and the decay rate r satisfy the equation $r\tau = \ln 2$.

14. Radium-226 has a half-life of 1620 years. Find the time period during which a given amount of this material is reduced by one-quarter.

15. According to Newton's law of cooling (see Problem 23 of Section 1.1), the temperature $u(t)$ of an object satisfies the differential equation

$$\frac{du}{dt} = -k(u - T),$$

    where T is the constant ambient temperature and k is a positive constant. Suppose that the initial temperature of the object is $u(0) = u_0$.
    (a) Find the temperature of the object at any time.

(b) Let τ be the time at which the initial temperature difference $u_0 - T$ has been reduced by half. Find the relation between k and τ.

16. Suppose that a building loses heat in accordance with Newton's law of cooling (see Problem 15) and that the rate constant k has the value $0.15\ \text{h}^{-1}$. Assume that the interior temperature is 70°F when the heating system fails. If the external temperature is 10°F, how long will it take for the interior temperature to fall to 32°F?

17. Consider an electric circuit containing a capacitor, resistor, and battery; see Figure 1.2.3. The charge $Q(t)$ on the capacitor satisfies the equation[5]

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where R is the resistance, C is the capacitance, and V is the constant voltage supplied by the battery.

(a) If $Q(0) = 0$, find $Q(t)$ at any time t, and sketch the graph of Q versus t.

(b) Find the limiting value Q_L that $Q(t)$ approaches after a long time.

(c) Suppose that $Q(t_1) = Q_L$ and that at time $t = t_1$ the battery is removed and the circuit is closed again. Find $Q(t)$ for $t > t_1$ and sketch its graph.

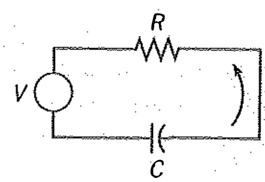

FIGURE 1.2.3 The electric circuit of Problem 17.

18. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 21 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

(a) Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

(b) Solve the problem in part (a) for $Q(t)$. How much chemical is in the pond after 1 year?

(c) At the end of 1 year the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

(d) Solve the initial value problem in part (c). How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

(e) How long does it take for $Q(t)$ to be reduced to 10 g?

(f) Plot $Q(t)$ versus t for 3 years.

19. Your swimming pool containing 60,000 gal of water has been contaminated by 5 kg of a nontoxic dye that leaves a swimmer's skin an unattractive green. The pool's filtering system can take water from the pool, remove the dye, and return the water to the pool at a flow rate of 200 gal/min.

[5] This equation results from Kirchhoff's laws, which are discussed in Section 3.7.

(a) Write down the initial value problem for the filtering process; let $q(t)$ be the amount of dye in the pool at any time t.

(b) Solve the problem in part (a).

(c) You have invited several dozen friends to a pool party that is scheduled to begin in 4 h. You have also determined that the effect of the dye is imperceptible if its concentration is less than 0.02 g/gal. Is your filtering system capable of reducing the dye concentration to this level within 4 h?

(d) Find the time T at which the concentration of dye first reaches the value 0.02 g/gal.

(e) Find the flow rate that is sufficient to achieve the concentration 0.02 g/gal within 4 h.

1.3 Classification of Differential Equations

The main purpose of this book is to discuss some of the properties of solutions of differential equations, and to present some of the methods that have proved effective in finding solutions or, in some cases, approximating them. To provide a framework for our presentation, we describe here several useful ways of classifying differential equations.

Ordinary and Partial Differential Equations. One important classification is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation.** In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation.**

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{1}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L; this equation is derived in Section 3.7. Typical examples of partial differential equations are the heat conduction equation

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{2}$$

and the wave equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{3}$$

Here, α^2 and a^2 are certain physical constants. Note that in both Eqs. (2) and (3) the dependent variable u depends on the two independent variables x and t. The heat conduction equation describes the conduction of heat in a solid body, and the wave equation arises in a variety of problems involving wave motion in solids or fluids.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example, the Lotka–Volterra, or predator–prey, equations are important in ecological modeling. They have the form

$$\begin{aligned} dx/dt &= ax - \alpha xy \\ dy/dt &= -cy + \gamma xy, \end{aligned} \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The constants a, α, c, and γ are based on empirical observations and depend on the particular species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka–Volterra equations are examined in Section 9.5. In some areas of application it is not unusual to encounter very large systems containing hundreds, or even many thousands, of equations.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first order equations, whereas Eq. (1) is a second order equation. Equations (2) and (3) are second order partial differential equations. More generally, the equation

$$F[t, u(t), u'(t), \ldots, u^{(n)}(t)] = 0 \tag{5}$$

is an ordinary differential equation of the nth order. Equation (5) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u', u'', \ldots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \ldots, y^{(n)}$ standing for $u'(t), u''(t), \ldots, u^{(n)}(t)$. Thus Eq. (5) is written as

$$F(t, y, y', \ldots, y^{(n)}) = 0. \tag{6}$$

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{7}$$

is a third order differential equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}). \tag{8}$$

This is mainly to avoid the ambiguity that may arise because a single equation of the form (6) may correspond to several equations of the form (8). For example, the equation

$$(y')^2 + ty' + 4y = 0 \tag{9}$$

leads to the two equations

$$y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}. \tag{10}$$

Linear and Nonlinear Equations. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$F(t, y, y', \ldots, y^{(n)}) = 0$$

is said to be **linear** if F is a linear function of the variables $y, y', \ldots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \tag{11}$$

Most of the equations you have seen thus far in this book are linear; examples are the equations in Sections 1.1 and 1.2 describing the falling object and the field mouse population. Similarly, in this section, Eq. (1) is a linear ordinary differential equation and Eqs. (2) and (3) are linear partial differential equations. An equation that is not of the form (11) is a **nonlinear** equation. Equation (7) is nonlinear because of the term yy'. Similarly, each equation in the system (4) is nonlinear because of the terms that involve the product xy.

A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle θ that an oscillating pendulum of length L makes with the vertical direction (see Figure 1.3.1) satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0, \tag{12}$$

whose derivation is outlined in Problems 29 through 31. The presence of the term involving $\sin\theta$ makes Eq. (12) nonlinear.

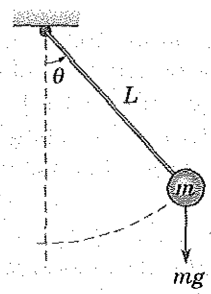

FIGURE 1.3.1 An oscillating pendulum.

The mathematical theory and methods for solving linear equations are highly developed. In contrast, for nonlinear equations the theory is more complicated, and methods of solution are less satisfactory. In view of this, it is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations. For example, for the pendulum, if the angle θ is small, then $\sin\theta \cong \theta$ and Eq. (12) can be approximated by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{13}$$

This process of approximating a nonlinear equation by a linear one is called **linearization**; it is an extremely valuable way to deal with nonlinear equations. Nevertheless, there are many physical phenomena that simply cannot be represented adequately by linear equations. To study these phenomena, it is essential to deal with nonlinear equations.

In an elementary text it is natural to emphasize the simpler and more straightforward parts of the subject. Therefore, the greater part of this book is devoted to linear equations and various methods for solving them. However, Chapters 8 and 9, as well as parts of Chapter 2, are concerned with nonlinear equations. Whenever it is appropriate, we point out why nonlinear equations are, in general, more difficult and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

Solutions. A **solution** of the ordinary differential equation (8) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \ldots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f[t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)] \tag{14}$$

for every t in $\alpha < t < \beta$. Unless stated otherwise, we assume that the function f of Eq. (8) is a real-valued function, and we are interested in obtaining real-valued solutions $y = \phi(t)$.

Recall that in Section 1.2 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

$$\frac{dp}{dt} = 0.5p - 450 \tag{15}$$

has the solution

$$p = 900 + ce^{t/2}, \tag{16}$$

where c is an arbitrary constant. It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a solution of a given equation, it is usually relatively easy to determine whether the function is actually a solution simply by substituting the function into the equation. For example, in this way it is easy to show that the function $y_1(t) = \cos t$ is a solution of

$$y'' + y = 0 \tag{17}$$

for all t. To confirm this, observe that $y_1'(t) = -\sin t$ and $y_1''(t) = -\cos t$; then it follows that $y_1''(t) + y_1(t) = 0$. In the same way you can easily show that $y_2(t) = \sin t$ is also a solution of Eq. (17). Of course, this does not constitute a satisfactory way to solve most differential equations, because there are far too many possible functions for you to have a good chance of finding the correct one by a random choice. Nevertheless, you should realize that you can verify whether any proposed solution is correct by substituting it into the differential equation. This can be a very useful check; it is one that you should make a habit of considering.

Some Important Questions. Although for the equations (15) and (17) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Does an equation of the form (8) always have a solution? The answer is "No." Merely writing down an equation of the form (8) does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of *existence* of a solution, and it is answered by theorems stating that under certain restrictions on the function f in Eq. (8), the equation always has solutions. This is not a purely mathematical concern for at least two reasons.

If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematically as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

If we assume that a given differential equation has at least one solution, then we may need to consider how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of *uniqueness*. In general, solutions of differential equations contain one or more arbitrary constants of integration, as does the solution (16) of Eq. (15). Equation (16) represents an infinity of functions corresponding to the infinity of possible choices of the constant c. As we saw in Section 1.2, if p is specified at some time t, this condition will determine a value for c; even so, we have not yet ruled out the possibility that there may be other solutions of Eq. (15) that also have the prescribed value of p at the prescribed time t. As in the question of existence of solutions, the issue of uniqueness has practical as well as theoretical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

A third important question is: Given a differential equation of the form (8), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, we discuss both elementary methods that can be used to obtain exact solutions of certain relatively simple problems, and also methods of a more general nature that can be used to find approximations to solutions of more difficult problems.

Computer Use in Differential Equations. A computer can be an extremely valuable tool in the study of differential equations. For many years computers have been used to execute numerical algorithms, such as those described in Chapter 8, to construct numerical approximations to solutions of differential equations. These algorithms have been refined to an extremely high level of generality and efficiency. A few lines of computer code, written in a high-level programming language and executed (often within a few seconds) on a relatively inexpensive computer, suffice to approximate to a high degree of accuracy the solutions of a wide range of differential equations. More sophisticated routines are also readily available. These routines combine the ability to handle very large and complicated systems with numerous diagnostic features that alert the user to possible problems as they are encountered.

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. There are on the market several well-crafted and relatively inexpensive special-purpose software packages for the graphical investigation of differential equations. The widespread availability of personal computers has brought powerful computational and graphical capability within the reach of individual students. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple, Mathematica, and MATLAB, each of which can be used on various kinds of personal computers or workstations. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. Maple and Mathematica also have very extensive analytical capabilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to delegate as many as possible of the routine (often repetitive) details to a computer, while you focus on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require a calculator or computer. Good judgment is often needed in selecting an effective combination.

PROBLEMS

In each of Problems 1 through 6, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2 \dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin t$

2. $(1 + y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$

3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$

4. $\dfrac{dy}{dt} + ty^2 = 0$

5. $\dfrac{d^2y}{dt^2} + \sin(t + y) = \sin t$

6. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + (\cos^2 t)y = t^3$

In each of Problems 7 through 14, verify that each given function is a solution of the differential equation.

7. $y'' - y = 0; \qquad y_1(t) = e^t, \quad y_2(t) = \cosh t$

8. $y'' + 2y' - 3y = 0; \quad y_1(t) = e^{-3t}, \quad y_2(t) = e^t$
9. $ty' - y = t^2; \quad y = 3t + t^2$
10. $y'''' + 4y''' + 3y = t; \quad y_1(t) = t/3, \quad y_2(t) = e^{-t} + t/3$
11. $2t^2y'' + 3ty' - y = 0, \quad t > 0; \quad y_1(t) = t^{1/2}, \quad y_2(t) = t^{-1}$
12. $t^2y'' + 5ty' + 4y = 0, \quad t > 0; \quad y_1(t) = t^{-2}, \quad y_2(t) = t^{-2}\ln t$
13. $y'' + y = \sec t, \quad 0 < t < \pi/2; \quad y = (\cos t)\ln\cos t + t\sin t$
14. $y' - 2ty = 1; \quad y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 15 through 18, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

15. $y' + 2y = 0$
16. $y'' - y = 0$
17. $y'' + y' - 6y = 0$
18. $y''' - 3y'' + 2y' = 0$

In each of Problems 19 and 20, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

19. $t^2y'' + 4ty' + 2y = 0$
20. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 21 through 24, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

21. $u_{xx} + u_{yy} + u_{zz} = 0$
22. $u_{xx} + u_{yy} + uu_x + uu_y + u = 0$
23. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
24. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 25 through 28, verify that each given function is a solution of the given partial differential equation.

25. $u_{xx} + u_{yy} = 0; \quad u_1(x,y) = \cos x\cosh y, \quad u_2(x,y) = \ln(x^2 + y^2)$
26. $\alpha^2u_{xx} = u_t; \quad u_1(x,t) = e^{-\alpha^2 t}\sin x, \quad u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin\lambda x, \quad \lambda$ a real constant
27. $a^2u_{xx} = u_{tt}; \quad u_1(x,t) = \sin\lambda x\sin\lambda at, \quad u_2(x,t) = \sin(x - at), \quad \lambda$ a real constant
28. $\alpha^2u_{xx} = u_t; \quad u = (\pi/t)^{1/2}e^{-x^2/4\alpha^2 t}, \quad t > 0$
29. Follow the steps indicated here to derive the equation of motion of a pendulum, Eq. (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.
    (a) Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.
    (b) Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.
    (c) Simplify the result from part (b) to obtain Eq. (12) in the text.
30. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.
    (a) Show that the kinetic energy T of the pendulum in motion is

$$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

    (b) Show that the potential energy V of the pendulum, relative to its rest position, is

$$V = mgL(1 - \cos\theta).$$

(c) By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate dE/dt, set it equal to zero, and show that the resulting equation reduces to Eq. (12).

31. A third derivation of the pendulum equation depends on the principle of angular momentum: The rate of change of angular momentum about any point is equal to the net external moment about the same point.

(a) Show that the angular momentum M, or moment of momentum, about the point of support is given by $M = mL^2 d\theta/dt$.

(b) Set dM/dt equal to the moment of the gravitational force, and show that the resulting equation reduces to Eq. (12). Note that positive moments are counterclockwise.

1.4 Historical Remarks

Without knowing something about differential equations and methods of solving them, it is difficult to appreciate the history of this important branch of mathematics. Further, the development of differential equations is intimately interwoven with the general development of mathematics and cannot be separated from it. Nevertheless, to provide some historical perspective, we indicate here some of the major trends in the history of the subject and identify the most prominent early contributors. Other historical information is contained in footnotes scattered throughout the book and in the references listed at the end of the chapter.

The subject of differential equations originated in the study of calculus by Isaac Newton (1642–1727) and Gottfried Wilhelm Leibniz (1646–1716) in the seventeenth century. Newton grew up in the English countryside, was educated at Trinity College, Cambridge, and became Lucasian Professor of Mathematics there in 1669. His epochal discoveries of calculus and of the fundamental laws of mechanics date from 1665. They were circulated privately among his friends, but Newton was extremely sensitive to criticism and did not begin to publish his results until 1687 with the appearance of his most famous book, *Philosophiae Naturalis Principia Mathematica*. Although Newton did relatively little work in differential equations as such, his development of the calculus and elucidation of the basic principles of mechanics provided a basis for their applications in the eighteenth century, most notably by Euler. Newton classified first order differential equations according to the three forms $dy/dx = f(x)$, $dy/dx = f(y)$, and $dy/dx = f(x,y)$. For the latter equation he developed a method of solution using infinite series when $f(x,y)$ is a polynomial in x and y. Newton's active research in mathematics ended in the early 1690s, except for the solution of occasional "challenge problems" and the revision and publication of results obtained much earlier. He was appointed Warden of the British Mint in 1696 and resigned his professorship a few years later. He was knighted in 1705 and, upon his death, was buried in Westminster Abbey.

Leibniz was born in Leipzig and completed his doctorate in philosophy at the age of 20 at the University of Altdorf. Throughout his life he engaged in scholarly work in several different fields. He was mainly self-taught in mathematics, since his interest in this subject developed when he was in his twenties. Leibniz arrived at the fundamental results of calculus independently, although a little later than Newton, but was the first to publish them, in 1684. Leibniz was very conscious of the power of good mathematical notation and was responsible for the notation dy/dx for the derivative and for the integral sign. He discovered the method of separation of variables (Section 2.2) in 1691, the reduction of homogeneous equations to separable ones (Section 2.2, Problem 30) in 1691, and the procedure for solving first order linear equations (Section 2.1) in 1694. He spent his life as ambassador and adviser to several German royal families, which permitted him to travel widely and to carry on an extensive correspondence with other mathematicians, especially the Bernoulli brothers. In the course of this correspondence many problems in differential equations were solved during the latter part of the seventeenth century.

The brothers Jakob (1654–1705) and Johann (1667–1748) Bernoulli of Basel did much to develop methods of solving differential equations and to extend the range of their applications. Jakob became professor of mathematics at Basel in 1687, and Johann was appointed to the same position upon his brother's death in 1705. Both men were quarrelsome, jealous, and frequently embroiled in disputes, especially with each other. Nevertheless, both also made significant contributions to several areas of mathematics. With the aid of calculus, they solved a number of problems in mechanics by formulating them as differential equations. For example, Jakob Bernoulli solved the differential equation $y' = [a^3/(b^2y - a^3)]^{1/2}$ in 1690 and, in the same paper, first used the term "integral" in the modern sense. In 1694 Johann Bernoulli was able to solve the equation $dy/dx = y/ax$. One problem which both brothers solved, and which led to much friction between them, was the *brachistochrone* problem (see Problem 32 of Section 2.3). The brachistochrone problem was also solved by Leibniz, Newton, and the Marquis de L'Hôpital. It is said, perhaps apocryphally, that Newton learned of the problem late in the afternoon of a tiring day at the Mint and solved it that evening after dinner. He published the solution anonymously, but upon seeing it, Johann Bernoulli exclaimed, "Ah, I know the lion by his paw."

Daniel Bernoulli (1700–1782), son of Johann, migrated to St. Petersburg as a young man to join the newly established St. Petersburg Academy but returned to Basel in 1733 as professor of botany and, later, of physics. His interests were primarily in partial differential equations and their applications. For instance, it is his name that is associated with the Bernoulli equation in fluid mechanics. He was also the first to encounter the functions that a century later became known as Bessel functions (Section 5.7).

The greatest mathematician of the eighteenth century, Leonhard Euler (1707–1783), grew up near Basel and was a student of Johann Bernoulli. He followed his friend Daniel Bernoulli to St. Petersburg in 1727. For the remainder of his life he was associated with the St. Petersburg Academy (1727–1741 and 1766–1783) and the Berlin Academy (1741–1766). Euler was the most prolific mathematician of all time; his collected works fill more than 70 large volumes. His interests ranged over all areas of mathematics and many fields of application. Even though he was blind during the last 17 years of his life, his work continued undiminished until the very day of his death. Of particular interest here is his formulation of problems in mechanics in mathematical language and his development of methods of solving these mathematical problems. Lagrange said of Euler's work in mechanics, "The first great work in which analysis is applied to the science of movement." Among other things, Euler identified the condition for exactness of first order differential equations (Section 2.6) in 1734–35, developed the theory of integrating factors (Section 2.6) in the same paper, and gave the general solution of homogeneous linear equations with constant coefficients (Sections 3.1, 3.3, 3.4, and 4.2) in 1743. He extended the latter results to nonhomogeneous equations in 1750–51. Beginning about 1750, Euler made frequent use of power series (Chapter 5) in solving differential equations. He also proposed a numerical procedure (Sections 2.7 and 8.1) in 1768–69, made important contributions in partial differential equations, and gave the first systematic treatment of the calculus of variations.

Joseph-Louis Lagrange (1736–1813) became professor of mathematics in his native Turin at the age of 19. He succeeded Euler in the chair of mathematics at the Berlin Academy in 1766 and moved on to the Paris Academy in 1787. He is most famous for his monumental work *Mécanique analytique*, published in 1788, an elegant and comprehensive treatise of Newtonian mechanics. With respect to elementary differential equations, Lagrange showed in 1762–65 that the general solution of an nth order linear homogeneous differential equation is a linear combination of n independent solutions (Sections 3.2 and 4.1). Later, in 1774–75, he gave a complete development of the method of variation of parameters (Sections 3.6 and 4.4). Lagrange is also known for fundamental work in partial differential equations and the calculus of variations.

Pierre-Simon de Laplace (1749–1827) lived in Normandy as a boy but came to Paris in 1768 and quickly made his mark in scientific circles, winning election to the Académie des Sciences in 1773. He was preeminent in the field of celestial mechanics; his greatest work, *Traité de mécanique céleste*, was published in five volumes between 1799 and 1825. Laplace's equation is fundamental in many branches of mathematical physics, and Laplace studied it extensively in connection with gravitational attraction. The Laplace transform (Chapter 6) is also named for him, although its usefulness in solving differential equations was not recognized until much later.

By the end of the eighteenth century many elementary methods of solving ordinary differential equations had been discovered. In the nineteenth century interest turned more toward the investigation of theoretical questions of existence and uniqueness and to the development of less elementary methods such as those based on power series expansions (see Chapter 5). These methods find their natural setting in the complex plane. Consequently, they benefitted from, and to some extent stimulated, the more or less simultaneous development of the theory of complex analytic functions. Partial differential equations also began to be studied intensively, as their crucial role in mathematical physics became clear. In this connection a number of functions, arising as solutions of certain ordinary differential equations, occurred repeatedly and were studied exhaustively. Known collectively as higher transcendental functions, many of them are associated with the names of mathematicians, including Bessel, Legendre, Hermite, Chebyshev, and Hankel, among others.

The numerous differential equations that resisted solution by analytical means led to the investigation of methods of numerical approximation (see Chapter 8). By 1900 fairly effective numerical integration methods had been devised, but their implementation was severely restricted by the need to execute the computations by hand or with very primitive computing equipment. In the last 60 years the development of increasingly powerful and versatile computers has vastly enlarged the range of problems that can be investigated effectively by numerical methods. Extremely refined and robust numerical integrators were developed during the same period and are readily available. Versions appropriate for personal computers have brought the ability to solve a great many significant problems within the reach of individual students.

Another characteristic of differential equations in the twentieth century was the creation of geometrical or topological methods, especially for nonlinear equations. The goal is to understand at least the qualitative behavior of solutions from a geometrical, as well as from an analytical, point of view. If more detailed information is needed, it can usually be obtained by using numerical approximations. An introduction to geometrical methods appears in Chapter 9.

Within the past few years these two trends have come together. Computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Unexpected phenomena (Section 9.8), such as strange attractors, chaos, and fractals, have been discovered, are being intensively studied, and are leading to important new insights in a variety of applications. Although it is an old subject about which much is known, the study of differential equations in the twenty-first century remains a fertile source of fascinating and important unsolved problems.

REFERENCES

Computer software for differential equations changes too fast for particulars to be given in a book such as this. A Google search for Maple, Mathematica, Sage, or MATLAB is a good way to begin if you need information about one of these computer algebra and numerical systems.

There are many instructional books on computer algebra systems, such as the following:

Cheung, C.-K., Keough, G. E., Gross, R. H., and Landraitis, C., *Getting Started with Mathematica* (3rd ed.) (New York: Wiley, 2009).

Meade, D. B., May, M., Cheung, C.-K., and Keough, G. E., *Getting Started with Maple* (3rd ed.) (New York: Wiley, 2009).

For further reading in the history of mathematics, see books such as those listed below:

Boyer, C. B., and Merzbach, U. C., *A History of Mathematics* (2nd ed.) (New York: Wiley, 1989).

Kline, M., *Mathematical Thought from Ancient to Modern Times* (3 vols.) (New York: Oxford University Press, 1990).

A useful historical appendix on the early development of differential equations appears in

Ince, E. L., *Ordinary Differential Equations* (London: Longmans, Green, 1927; New York: Dover, 1956).

Encyclopedic sources of information about the lives and achievements of mathematicians of the past are

Gillespie, C. C., ed., *Dictionary of Scientific Biography* (15 vols.) (New York: Scribner's, 1971).

Koertge, N., ed., *New Dictionary of Scientific Biography* (8 vols.) (New York: Scribner's, 2007).

Koertge, N., ed., *Complete Dictionary of Scientific Biography* (New York: Scribner's, 2007 [e-book]).

Much historical information can be found on the Internet. One excellent site is the MacTutor History of Mathematics archive

http://www-history.mcs.st-and.ac.uk/history/

created by John J. O'Connor and Edmund F. Robertson, Department of Mathematics and Statistics, University of St. Andrews, Scotland.