Lesson 34

Function:

- a connection (relationship) between sets in which each element of the first set corresponds with exactly one element of the second set (one and only one)
- the first set is called the **domain** and the second set is called the **range**
- every input (element of domain) can only have one output (element of range)
- the graph of a function must pass the vertical line test
    - a vertical line can only intersect the graph of a function at once
- functions may be represented using listings, graphs, or equations

Example 1a: If $f(x) = x^2$, find

a. $f(2)$

b. $f(-2)$

Example 1b: If $g(x) = x^3$, find

a. $g(2)$

b. $g(-2)$

Notice that for the function f, both inputs (2 and -2) result in the same output (4); f is a function because each input results in exactly one output; however it is **not a one-to-one function** because each output is not the result of exactly one input. The output 4 is the result of two different inputs, 2 and -2. Notice that for the function g, both inputs (2 and -2) result in different outputs. Function g is **a one-to-one function** because each ouput is the result of exactly one input.

One-to-one function:

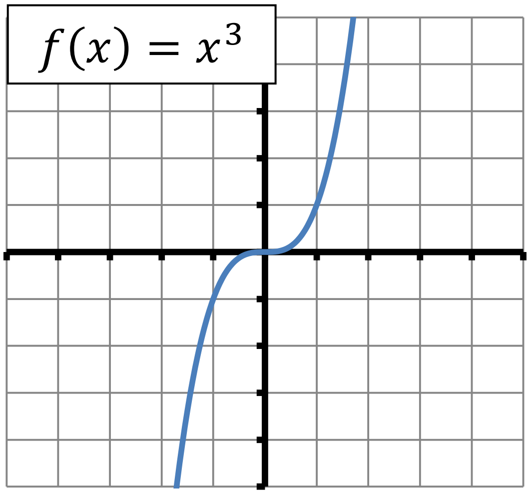
- a function in which each member of the range corresponds to exactly one member (one and only one member) of the domain
- every input has exactly one output (this makes it a function) and every output is the result of exactly one input (this makes it one-to-one)
- A 1-1 function is **always increasing** or **always decreasing**.

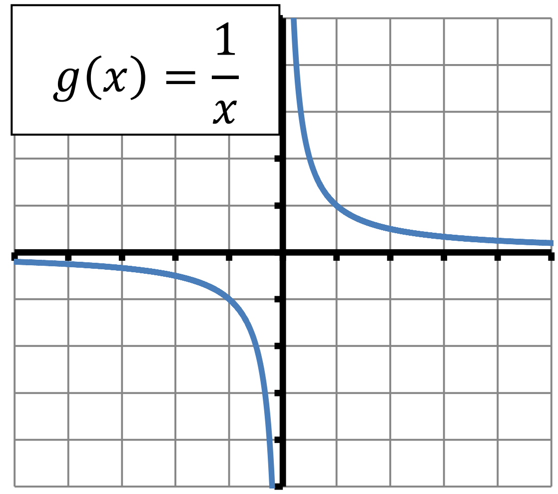
To determine **whether a function is one-to-one based on its graph**, we can use the **horizontal line test.**

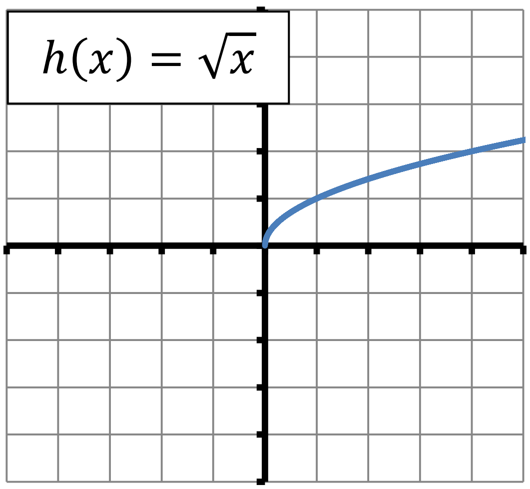
Horizontal line test:

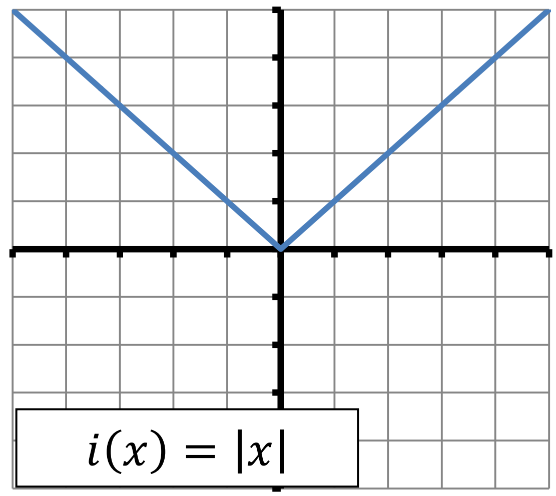
- If any horizontal line intersects the graph of a function at one and only one point, the function is one-to-one.
- If any horizontal line intersects the graph of a function at more than one point, the function is **not** one-to-one.

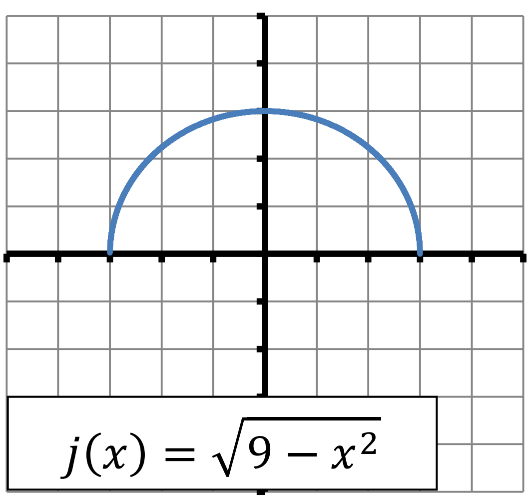
<u>Example 2:</u> Given the graphs of the following functions, determine which functions are one-to-one.

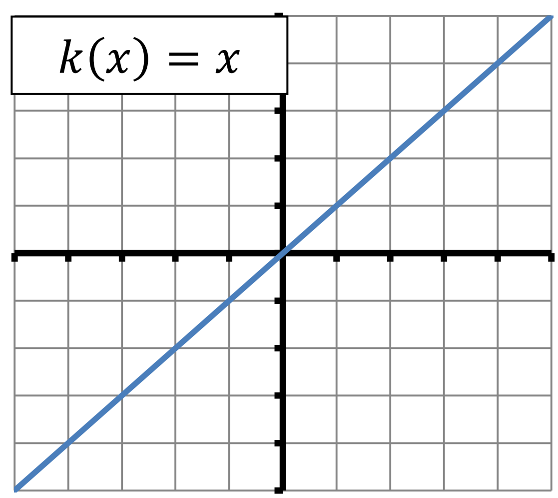












Theorem on Increasing/Decreasing Functions: A function that is strictly increasing or strictly decreasing throughout its domain is one-to-one. You will notice from the example 2 graphs that this is another way to determine if a function is 1-1.

Example 3: Determine which of the following functions are one-to-one using any method. Hint: Compare pairs of points of form $(x, y_1) and\ (x, y_2)$ or make a sketch (graph) of each.

a. $f(x) = -3x + 1$ (a linear function)

b. $g(x) = x^2 - 9$ (a quadratic function, parabola opening upward)

c. $j(x) = -|x|$ (an absolute value function, V shape)

d. $k(x) = \sqrt{4 - x^2}$ (upper half-circle)

e. $m(x) = -x^3 + 3$

The reason we need to determine whether a function is one-to-one is because only a one-to-one function can have an inverse. An inverse is the function that 'undoes' some original function.
One way to find an inverse function is to solve for the 'other' variable. This method can often be used to find the inverse of a formula.

Example 4a: The linear function $F(C) = \frac{9}{5}C + 32$ expresses the temperature in Fahrenheit degrees (F) based on the temperature in Celsius degrees (C).

a. Find $F(10)$; what does this represent?

b. Find the inverse function $\big(C(F)\big)$; what does this function represent?

c. Find $C(50)$; what does this represent?

Example 4b: The function $A(r) = \pi r^2$ (for $r > 0$) expresses the area of a circle (A) based on the length of the radius of the circle (r).

a. Find $A(2)$; what does this represent? Write the ordered pair.

b. Find the inverse function ($r(A)$); what does this function represent?

c. Find $r(4\pi)$; what does this represent? Write the ordered pair.

Inverse function:

- denoted symbolically by f^{-1}
  - f^{-1} means the inverse of the function f
  - f^{-1} does **NOT** mean $\frac{1}{f}$
- the function that undoes some original function
  - $f(x) = x + 3;\ ex.\ (5, 8)$ $\quad f^{-1}(x) = x - 3;\ ex. (8, 5)$
  - $g(x) = -4x;\ ex. (2, -8)$ $\quad g^{-1}(x) = \frac{x}{-4};\ ex.\ (-8, 2)$
  - $h(x) = 2x - 5; ex. (1, -3)$ $\quad h^{-1}(x) = \frac{1}{2}x + \frac{5}{2}; ex.\ (-3, 1)$
- to obtain an inverse function, switch the inputs and outputs of an existing function (switch the *x* and *y* variables)
- inverse functions only exist for one-to-one functions; why?

Below are some examples of inverse functions.

Note: The inverse notation f^{-1} does not indicate 'reciprocal' as we typically denote. In this context, it means the inverse function.

Example A: $f(x) = x + 3$; ordered pair (5, 8)

$f^{-1}(x) = x - 3$; ordered pair (8, 5)

(continued on next page)

Example B: $g(x) = -4x$; ordered pair (2, −8)
$g^{-1}(x) = -\frac{x}{4}$; ordered pair (−8, 2)

Example C: $h(x) = 2x - 5$; ordered pair $(1, -3)$
$h^{-1}(x) = \frac{x+5}{2}$; ordered pair $(-3, 1)$

Finding the inverse of a function in variables *x* and *y*:

1. Change the function to an equation by replacing $f(x)$ with y.
2. Switch the inputs and outputs (*x* and *y* variables).
3. Solve for y.
4. Once y is isolated, replace y with $f^{-1}(x)$, the inverse notation.

Example 5: Find the inverse of each of function. Find an ordered pair in the inverse function that corresponds to an ordered pair in the original function.

a. $f(x) = \frac{1}{x+3}$

b. $f(x) = \frac{9x+2}{2x-7}$

c. $f(x) = \frac{9x}{x-6}$

$d.$ $f(x) = \sqrt{x-2}$

$e.$ $g(x) = x^2 - 2, \quad x \leq 0$

$f.$ $h(x) = -\sqrt{x+4}$

Example 5 ½: For each function, find (a) its domain and range, (b) its inverse, and (c) the domain and range of the inverse function.

$a)\ \ f(x) = \sqrt{x-2}$ $\qquad\qquad$ $b)\ \ g(x) = x^2 - 2,\ x \le 0$

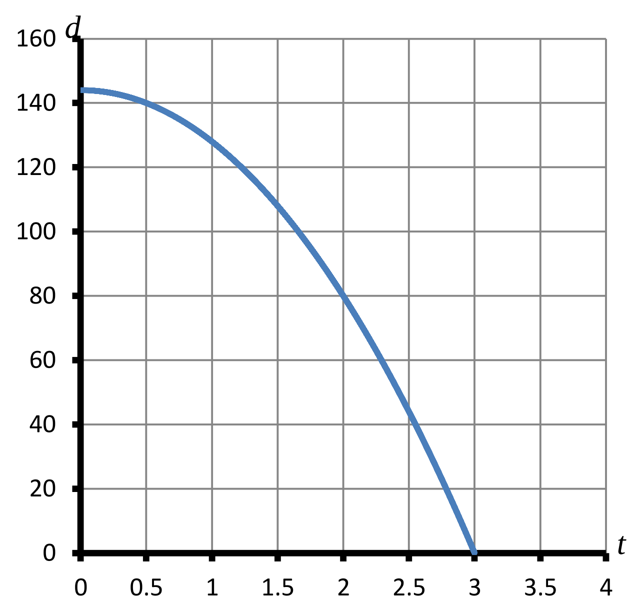
$c)\ \ h(x) = -\sqrt{x+4}$

Example 6: When an object is dropped from the top of a 144 foot building, the distance of the object above the ground d (in feet) after t seconds is given by the function $d(t) = -16t^2 + 144$. (graph below)

a. What is the domain and range of the function d?

b. Find the inverse function of d; what does this function represent?
$d^{-1}(t) = t(d) =$

c. What is the domain and range of the inverse function?

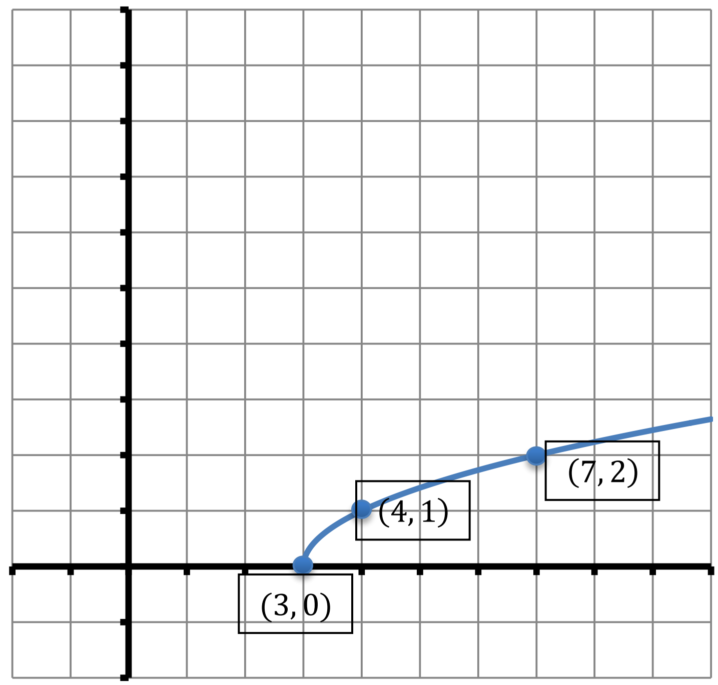


Domain and Range of f and f^{-1}:

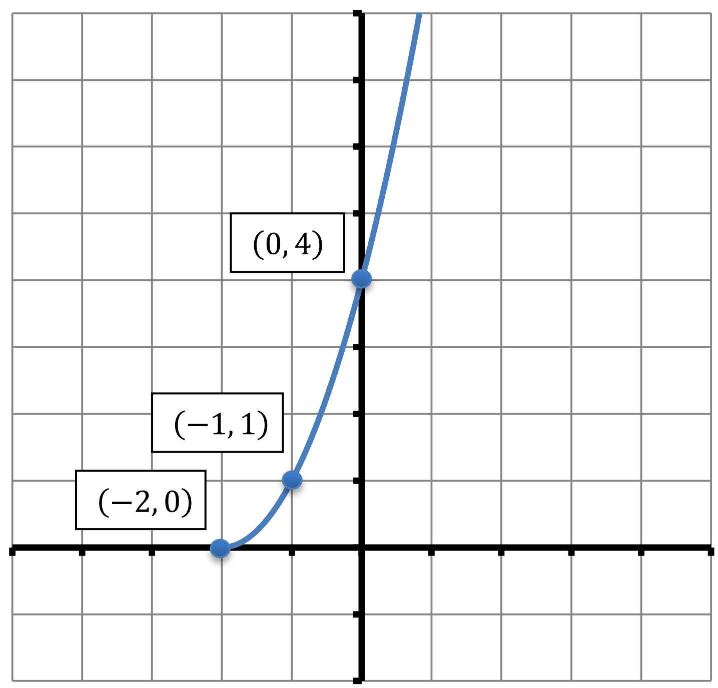
- the domain of f is the range of f^{-1}
- the range of f is the domain of f^{-1}
- the domain and range of f and f^{-1} are switched because the inputs and outputs are switched

Example 7: List the domain and range of each of the following functions. Then find and graph the inverse function and list its domain and range.

a. $f(x) = \sqrt{x-3}$ (graph at right)



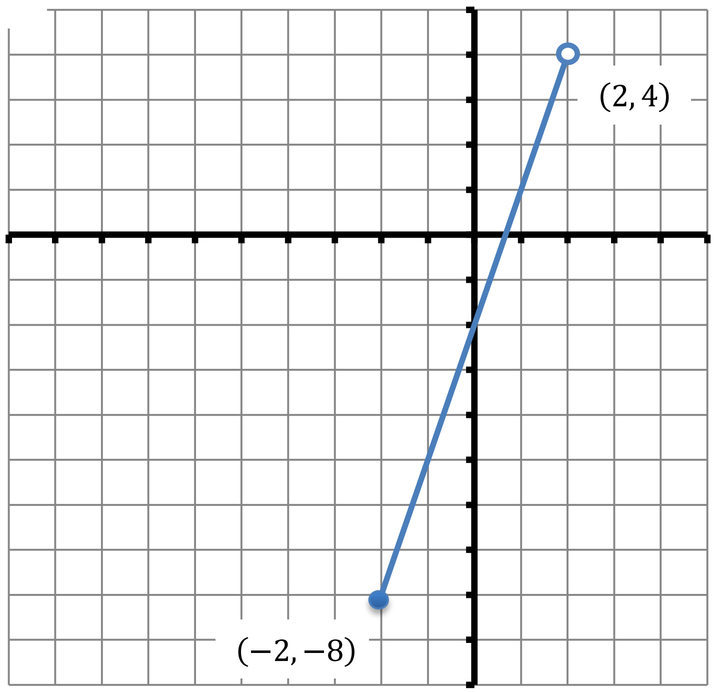
b. $f(x) = (x+2)^2, x \geq -2$ (graph at right)



The graph of a function and its inverse will always be symmetric about the line $y = x$.

Example 8: The graphs of one-to-one functions f and g are shown. Sketch the graphs of f^{-1} and g^{-1}, then find the domain and range of all the functions.

f



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