In lesson 9 (previous lesson), we determined a few ways to find a limit \( \lim_{x \to a} f(x) \).

- Sometimes, especially with polynomial expressions or polynomial functions, direct substitution could be used; the limit was the value when \( a \) was evaluated in \( f(x) \).
- Sometimes we could use a table (or a graph) and select values closer and closer to \( a \), from the left and from the right to see if the approached values were the same.
- Sometime we could simplify the expression \( f(x) \) and cancel common factors and then use direct substitution.

Here is a summary.

**Techniques for Evaluating Limits:**

1. With a polynomial function (or many other functions), **direct substitution sometimes can be used to find the limit.** See the examples below.
   
   \( a) \lim_{x \to (-2)} (x^2 - x) = (-2)^2 - (-2) = 4 + 2 = 6 \)
   
   \( b) \lim_{n \to 3} (2n - 4) = 2(3) - 4 = 6 - 4 = 2 \)
   
   \( c) \lim_{a \to 5} \sqrt{2a + 6} = \sqrt{2(5) + 6} = \sqrt{16} = 4 \)
   
   \( d) \lim_{n \to 2} \left( \frac{2n - 5}{n + 1} \right) = \frac{2(2) - 5}{2 + 1} = -1 \) or \(-\frac{1}{3} \)

2. With a rational function or rational expression, you can **sometimes write an equivalent expression for the function by simplifying**, then use ‘direct substitution’ in the equivalent expression. Direct substitution before simplifying may yield 0/0.

   \( a) \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \left( \frac{(x + 3)(x - 3)}{x - 3} \right) = \lim_{x \to 3} (x + 3) = 3 + 3 = 6 \)
   
   \( b) \lim_{c \to 5} \left( \frac{c^2 - 4c - 5}{c - 5} \right) = \lim_{c \to 5} \left( \frac{(c - 5)(c + 1)}{c - 5} \right) = \lim_{c \to 5} (c + 1) = 5 + 1 = 6 \)
   
   \( c) \lim_{\Delta x \to 0} \left( \frac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} \right) = \lim_{\Delta x \to 0} \left( \frac{3x + 3(\Delta x) - 2 - 3x + 2}{\Delta x} \right) = \lim_{\Delta x \to 0} \left( \frac{3(\Delta x)}{\Delta x} \right) = \lim_{\Delta x \to 0} 3 = 3 \)

In this lesson we will discover a few other techniques to find limits.

We have ‘rationalized denominators’ to simplify an expression with a square root in the denominator. Sometimes this was done by multiplying numerator and denominator by the **conjugate** of the denominator as in this expression.

\[
\text{Simplify: } \frac{2}{\sqrt{5} + 3} = \frac{2(\sqrt{5} - 3)}{(\sqrt{5} + 3)(\sqrt{5} - 3)} = \frac{2(\sqrt{5} - 3)}{5 - 9} = \frac{2(\sqrt{5} - 3)}{-4} = \frac{-(\sqrt{5} - 3)}{2} \text{ or } \frac{3}{2} \cdot \frac{-\sqrt{5}}{2}
\]
When finding limits and direct substitution lead to the indeterminant form, it is sometimes beneficial to rationalize the numerator. Examine this example:

Ex 1: Find the limit:
\[
\lim_{x \to 16} \frac{\sqrt{x} - 4}{x - 16}
\]
\[
= \lim_{x \to 16} \frac{(\sqrt{x} - 4)(\sqrt{x} + 4)}{(x - 16)(\sqrt{x} + 4)}
\]
Numerator and Denominator were multiplied by \((\sqrt{x} + 4)\).
\[
= \lim_{x \to 16} \frac{x - 16}{(x - 16)(\sqrt{x} + 4)}
\]
Cancel the common factor
\[
= \lim_{x \to 16} \frac{1}{\sqrt{x} + 4}
\]
Direct substitution.
\[
= \frac{1}{\sqrt{16} + 4} = \frac{1}{4 + 4} = \frac{1}{8}
\]

Techniques for Evaluating Limits (continued):

3. With a limit of the form \(\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}\), try rationalizing the numerator.

Ex. 2: Find each limit.
\[
a) \quad \lim_{x \to 144} \frac{12 - \sqrt{x}}{144 - x}
\]
\[
b) \quad \lim_{a \to 4} \frac{\sqrt{a} - 2}{a - 4}
\]

Basic limit at infinity principle: For any positive real number \(n\),
\[
\lim_{x \to \infty} \left( \frac{1}{x^n} \right) = 0 \quad \text{and} \quad \lim_{x \to -\infty} \left( \frac{1}{x^n} \right) = 0 \quad \text{for positive values of} \ x \ \text{only in the 2nd case}.
\]
(Limits at infinity correspond to horizontal asymptotes of the graph of a function.)

Study this example:

\[
\lim_{x \to \infty} \left( \frac{4x + 3}{5x - 2} \right) = \lim_{x \to \infty} \left( \frac{4x}{5x} - \frac{3}{x} \right) = \lim_{x \to \infty} \left( \frac{4 + \frac{3}{x}}{5 - \frac{2}{x}} \right) = \frac{4 + 0}{5} = \frac{4}{5}
\]

Direct substitution of \(\infty\) or \(-\infty\) does not yield a number or limit value. Sometimes you get \(\frac{\infty}{\infty}\) or \(\frac{-\infty}{\infty}\). Such expressions have no meaning. A technique described below must be used.

Techniques for Evaluating Limits (continued):

4. When finding the limit of a rational expression or rational function when the variable is approaching \(\pm \infty\), divide each term of the numerator and denominator by the highest power of the variable in the denominator, simplify and then evaluate the limit. Use the fact that \(\lim_{x \to \pm \infty} \left( \frac{1}{x^n} \right) = 0\) and \(\lim_{x \to \pm \infty} \left( \frac{1}{x^d} \right) = 0\) (basic limit at infinity principle).

Ex 3: Find each limit.

\[ a) \lim_{u \to \infty} \left( \frac{12u^2}{5u^2 - 3u + 1} \right) \]

\[ b) \lim_{x \to \infty} \left( \frac{-9x}{3x + 7} \right) \]

\[ c) \lim_{x \to \infty} \left( \frac{2x^2 - 4}{3x^2 + 4x^2 - x} \right) \]

\[ d) \lim_{c \to \infty} \left( \frac{4c^5 - 3c^3 + 2c}{5c^3 + 4c^4 - 6c} \right) \]
Example 4: This example is problem 84 on page 139 of the textbook. You will need to look at the graph of this problem.

The graph shows how the postage required to mail a letter in the U. S. has changed in recent years. Let $C(t)$ be the cost to mail a letter in the year $t$. Find the following. (You will have to approximate to the nearest cent.)

\[ a) \lim_{t \to 2009^-} C(t) \quad b) \lim_{t \to 2009^+} C(t) \]

\[ c) \lim_{t \to 2009} C(t) \quad d) C(2009) \]

e) \lim_{x \to \infty} \left( \frac{2x^2 - 4x + 3}{x - 5} \right) \quad f) \lim_{v \to -\infty} \left( \frac{5v^3 - 3v^4}{2v^3 - 3v + 2} \right) \]