MA 15910 Lesson 26 Notes
Graphs of Rational Functions (Asymptotes)
Limits at infinity

Definition of a Rational Function:
If $P(x)$ and $Q(x)$ are both polynomial functions, $Q(x) \neq 0$, then the function *f* below is called a **Rational Function.**

$$f(x) = \frac{P(x)}{Q(x)}$$

The following are examples of rational functions.

$$f(x) = \frac{2}{x}, \quad g(x) = \frac{x-5}{x+3}, \quad h(x) = \frac{(x+3)(x-3)(x+5)}{(x+6)^2}, \quad j(x) = \frac{x^5+3x^3}{2x^4-4x^3+5x-1}$$

If there are any values of *x* for which the denominator of the function would equal 0, these values must be excluded from the domain of the function. At those values of *x*, the function value does not exist or is not defined and the graph of the function is discontinuous at that value (has a break in the graph).

Any values of $Q(x)$ for which $Q(x) = 0$, but $P(x) \neq 0$, are values where there is a **vertical asymptote.** This is a vertical line where the graph of *f* (function values) approaches ∞ $or -\infty$ as *x* goes toward the number we will call *c* (any number that makes the denominator equal 0).. This is because $\lim_{x \to c} f(x) = \infty$ or $-\infty$. Since it is a vertical line, its equation is $x = c$.

Find the equation(s) of any vertical asymptotes for each rational function.

Ex. 1: $f(x) = \frac{12}{x-3}$

Ex. 2: $g(x) = \frac{x-5}{x^2+7x+10}$

Ex 3: $y = \frac{x^2-3x-10}{x^2+2x+1}$

Ex 4: $j(x) = \frac{x+2}{x^3-x^2-12x}$

Ex 5: $r(x) = \frac{x^2+7x-18}{(x+1)(x+9)}$

Note: In example 5 of the previous page, there was a factor that 'cancelled out' in both the numerator and denominator. That factor would not give a vertical asymptote. **IN FACT: THERE IS A HOLE IN THE GRAPH AT THE VALUE THAT MAKES THAT FACTOR BE ZERO.**

Find the following limits by dividing each term in the numerator and denominator by the highest power of x in the denominator. Also use the fact that the value of $\frac{1}{x^n}$ approaches 0 as x goes toward either ∞ or −∞. (As a denominator gets larger and larger, but the numerator stays relative smaller; the value of the fraction is almost zero.)

$\lim_{n\to\infty}\frac{1}{n}=0$ and $\lim_{n\to-\infty}\frac{1}{n}=0$ **This is the basic theorem for limits at infinity.**

Find the following limits, if they exist. If a limit goes to ∞ or −∞ (gets larger without bound or smaller without bound), you may say the limit equals to ∞ or −∞.

Ex 6:

$$\lim_{x\to\infty}\frac{3x^4-2x}{5x^4-7x^3+5x^2-x}=\lim_{x\to\infty}\frac{\frac{3x^4}{x^4}-\frac{2x}{x^4}}{\frac{5x^4}{x^4}-\frac{7x^3}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}}$$

$$=\lim_{x\to\infty}\frac{3-\frac{2x}{x^4}}{5-\frac{7x^3}{x^4}+\frac{5x^2}{x^4}-\frac{x}{x^4}}$$

$$=\frac{3-0}{5-0+0-0}$$

$$=\frac{3}{5}$$

Ex 7:

$$\lim_{a\to(-\infty)}\frac{2a-4a^2+9a^5}{2-3a^3-6a^5}$$

Ex 8:

$$\lim_{x\to\infty}\frac{2x-3}{x^2-5x+6}$$

Ex 9:

$$\lim_{x\to(-\infty)}\frac{5x^6+2x^4-3x^2}{5x^2-3x+2}$$

In examples 6 and 7 of the previous page, <u>the leading term of the numerator had the same degree as the leading term of the denominator</u>. The limit was the ratio of the coefficients of those leading terms.
In example 8 on the previous page, <u>the leading term of the numerator had a lesser degree than the leading term in the denominator</u>. The limit was zero.
In example 9 on the previous page, <u>the leading term of the numerator had a greater degree than the leading term in the denominator</u>. The limit was ∞.

****Finding a horizontal asymptote of a <u>rational</u> function (horizontal line that the graph approaches as *x* goes to ∞ or $-\infty$) uses the same procedure as the limits at $\pm\infty$ in examples 6 through 9. Here is an easy way to determine these limits or the equation of any horizontal asymptote.****

1. If the numerator has a lesser degree than the denominator, there is a horizontal asymptote with equation $y = 0$ (the *x*-axis).
2. If the numerator and denominator have the same degree, there is a horizontal asymptote with equation $y =$ the ratio of the leading coefficients.
3. If the numerator has a greater degree than the denominator, there is no horizontal asymptote.

****<u>Note</u>**: <u>We will **not** be discussing **oblique asymptotes** (lines other than vertical or horizontal that a graph would approach</u>; see example 7 on page 781 of text). One of the lines that the graph approaches is a 'slanted' line. **On any assignment problems in MyMathLab that ask for equations of oblique asymptotes, answer NO. You are not responsible to find any oblique asymptotes on homework problems (paper or MyMathLab).**

<u>Ex 10</u>: Find the equations of any horizontal or vertical asymptotes for these rational functions.

$a)\quad g(x) = \dfrac{12x^4 - 5x^2 + 7}{9x - 3x^4}$

$b)\quad F(x) = \dfrac{2x^3 - 4x}{5x^2 + 6x + 1}$

$c)\quad h(x) = \dfrac{9x + 3}{5x^2 - 2x}$

Note: Equations for horizontal asymptotes correspond to limits at infinity or negative infinity. Even though we do not usually describe a limit as ∞ (only a number), in MyMathLab, they may except ∞ as the answer for a limit.

<u>Ex 11:</u> Find the following limits. It a limit is 'unbounded', you may write ∞ or -∞ for the limit value.

$a)\quad \lim_{x\to\infty} \frac{2x-3}{7x+5}$

$b)\quad \lim_{n\to(-\infty)} \frac{4n^2-2n+9}{3n+5}$

$c)\quad \lim_{x\to\infty} \frac{2x^2-7x+7}{3x^3-5x^2+x}$

Optional: Sketch a graph of *f* using asymptotes, intercepts, and perhaps plotting a few points.

$$f(x) = \frac{6x+3}{2x-4}$$