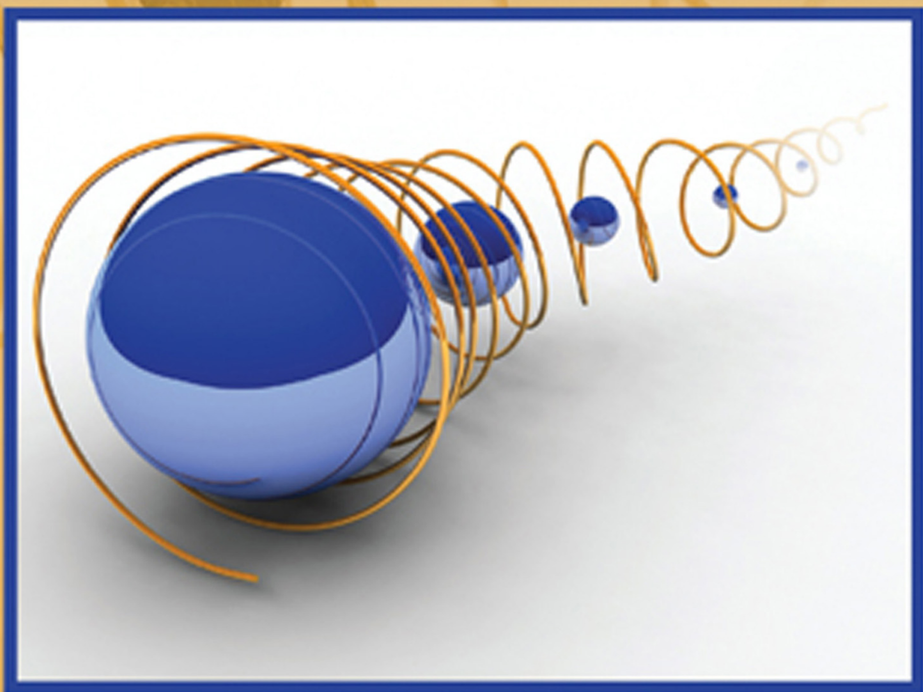
CLASSIC TWELFTH EDITION

ALGEBRA AND TRIGONOMETRY

WITH ANALYTIC GEOMETRY



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1

Fundamental Concepts of Algebra

1.1 Real Numbers

1.2 Exponents and Radicals

1.3 Algebraic Expressions

1.4 Fractional Expressions

The word *algebra* comes from *ilm al-jabr w'al muqabala,* the title of a book written in the ninth century by the Arabian mathematician al-Khworizimi. The title has been translated as the science of restoration and reduction, which means transposing and combining similar terms (of an equation). The Latin transliteration of al-jabr led to the name of the branch of mathematics we now call algebra.

In algebra we use symbols or letters—such as a, b, c, d, x, y—to denote arbitrary numbers. This general nature of algebra is illustrated by the many formulas used in science and industry. As you proceed through this text and go on either to more advanced courses in mathematics or to fields that employ mathematics, you will become more and more aware of the importance and the power of algebraic techniques.

1.1 Real Numbers

Real numbers are used throughout mathematics, and you should be acquainted with symbols that represent them, such as

$$1, \quad 73, \quad -5, \quad \tfrac{49}{12}, \quad \sqrt{2}, \quad 0, \quad \sqrt[3]{-85}, \quad 0.33333\ldots, \quad 596.25,$$

and so on. The **positive integers,** or **natural numbers,** are

$$1, \quad 2, \quad 3, \quad 4, \quad \ldots .$$

The **whole numbers** (or *nonnegative integers*) are the natural numbers combined with the number 0. The **integers** are often listed as follows:

$$\ldots, \quad -4, \quad -3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \quad 4, \quad \ldots$$

Throughout this text lowercase letters $a, b, c, x, y, \ldots$ represent arbitrary real numbers (also called *variables*). If a and b denote the same real number, we write $a = b$, which is read "a **is equal to** b" and is called an **equality**. The notation $a \neq b$ is read "a **is *not* equal to** b."

If a, b, and c are integers and $c = ab$, then a and b are **factors,** or **divisors,** of c. For example, since

$$6 = 2 \cdot 3 = (-2)(-3) = 1 \cdot 6 = (-1)(-6),$$

we know that $1, -1, 2, -2, 3, -3, 6$, and -6 are factors of 6.

A positive integer p different from 1 is **prime** if its only positive factors are 1 and p. The first few primes are 2, 3, 5, 7, 11, 13, 17, and 19. The **Fundamental Theorem of Arithmetic** states that every positive integer different from 1 can be expressed as a product of primes in one and only one way (except for order of factors). Some examples are

$$12 = 2 \cdot 2 \cdot 3, \quad 126 = 2 \cdot 3 \cdot 3 \cdot 7, \quad 540 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5.$$

A **rational number** is a real number that can be expressed in the form a/b, where a and b are integers and $b \neq 0$. Note that every integer a is a rational number, since it can be expressed in the form $a/1$. Every real number can be expressed as a decimal, and the decimal representations for rational numbers are either *terminating* or *nonterminating and repeating*. For example, we can show by using the arithmetic process of division that

$$\tfrac{5}{4} = 1.25 \quad \text{and} \quad \tfrac{177}{55} = 3.2181818\ldots,$$

where the digits 1 and 8 in the representation of $\frac{177}{55}$ repeat indefinitely (sometimes written $3.2\overline{18}$).

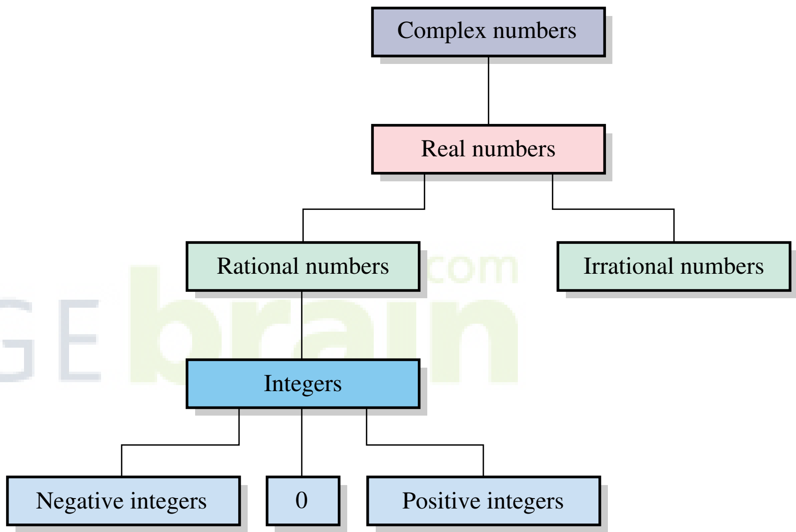
Real numbers that are not rational are **irrational numbers.** Decimal representations for irrational numbers are always *nonterminating and nonrepeating*. One common irrational number, denoted by π, is the ratio of the circumference of a circle to its diameter. We sometimes use the notation $\pi \approx 3.1416$ to indicate that π **is approximately equal to** 3.1416.

In technical writing, the use of the symbol $\doteq$ *for* ***is approximately equal to*** *is convenient.*

There is no *rational* number b such that $b^2 = 2$, where b^2 denotes $b \cdot b$. However, there is an *irrational* number, denoted by $\sqrt{2}$ (the **square root** of 2), such that $(\sqrt{2})^2 = 2$.

The system of **real numbers** consists of all rational and irrational numbers. Relationships among the types of numbers used in algebra are illustrated in the diagram in Figure 1, where a line connecting two rectangles means that the numbers named in the higher rectangle include those in the lower rectangle. The complex numbers, discussed in Section 2.4, contain all real numbers.

Figure 1 Types of numbers used in algebra



The real numbers are **closed relative to the operation of addition** (denoted by $+$); that is, to every pair a, b of real numbers there corresponds exactly one real number $a + b$ called the **sum** of a and b. The real numbers are also **closed relative to multiplication** (denoted by $\cdot$); that is, to every pair a, b of real numbers there corresponds exactly one real number $a \cdot b$ (also denoted by ab) called the **product** of a and b.

Important properties of addition and multiplication of real numbers are listed in the following chart.

Properties of Real Numbers

Terminology	General case	Meaning
(1) Addition is **commutative.**	$a + b = b + a$	Order is immaterial when adding two numbers.
(2) Addition is **associative.**	$a + (b + c) = (a + b) + c$	Grouping is immaterial when adding three numbers.
(3) 0 is the **additive identity.**	$a + 0 = a$	Adding 0 to any number yields the same number.
(4) $-a$ is the **additive inverse,** or **negative,** of a.	$a + (-a) = 0$	Adding a number and its negative yields 0.
(5) Multiplication is **commutative.**	$ab = ba$	Order is immaterial when multiplying two numbers.
(6) Multiplication is **associative.**	$a(bc) = (ab)c$	Grouping is immaterial when multiplying three numbers.
(7) 1 is the **multiplicative identity.**	$a \cdot 1 = a$	Multiplying any number by 1 yields the same number.
(8) If $a \neq 0$, $\dfrac{1}{a}$ is the **multiplicative inverse,** or **reciprocal,** of a.	$a\left(\dfrac{1}{a}\right) = 1$	Multiplying a nonzero number by its reciprocal yields 1.
(9) Multiplication is **distributive** over addition.	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$	Multiplying a number and a sum of two numbers is equivalent to multiplying each of the two numbers by the number and then adding the products.

Since $a + (b + c)$ and $(a + b) + c$ are always equal, we may use $a + b + c$ to denote this real number. We use abc for either $a(bc)$ or $(ab)c$. Similarly, if four or more real numbers a, b, c, d are added or multiplied, we may write $a + b + c + d$ for their sum and $abcd$ for their product, regardless of how the numbers are grouped or interchanged.

The distributive properties are useful for finding products of many types of expressions involving sums. The next example provides one illustration.

EXAMPLE 1 Using distributive properties

If p, q, r, and s denote real numbers, show that

$$(p + q)(r + s) = pr + ps + qr + qs.$$

SOLUTION We use both of the distributive properties listed in (9) of the preceding chart:

$$\begin{aligned} &(p + q)(r + s) \\ &\quad = p(r + s) + q(r + s) && \text{second distributive property, with } c = r + s \\ &\quad = (pr + ps) + (qr + qs) && \text{first distributive property} \\ &\quad = pr + ps + qr + qs && \text{remove parentheses} \end{aligned}$$

The following are basic properties of equality.

Properties of Equality	If $a = b$ and c is any real number, then **(1)** $a + c = b + c$ **(2)** $ac = bc$

Properties 1 and 2 state that the same number may be added to both sides of an equality, and both sides of an equality may be multiplied by the same number. We will use these properties extensively throughout the text to help find solutions of equations.

The next result can be proved.

Products Involving Zero	**(1)** $a \cdot 0 = 0$ for every real number a. **(2)** If $ab = 0$, then either $a = 0$ or $b = 0$.

When we use the word *or* as we do in (2), we mean that *at least* one of the factors a and b is 0. We will refer to (2) as the *zero factor theorem* in future work.

Some properties of negatives are listed in the following chart.

Properties of Negatives

Property	Illustration
(1) $-(-a) = a$	$-(-3) = 3$
(2) $(-a)b = -(ab) = a(-b)$	$(-2)3 = -(2 \cdot 3) = 2(-3)$
(3) $(-a)(-b) = ab$	$(-2)(-3) = 2 \cdot 3$
(4) $(-1)a = -a$	$(-1)3 = -3$

The reciprocal $\dfrac{1}{a}$ of a nonzero real number a is often denoted by a^{-1}, as in the next chart.

Notation for Reciprocals

Definition	Illustrations
If $a \neq 0$, then $a^{-1} = \dfrac{1}{a}$.	$2^{-1} = \dfrac{1}{2}$ $\left(\dfrac{3}{4}\right)^{-1} = \dfrac{1}{3/4} = \dfrac{4}{3}$

Note that if $a \neq 0$, then

$$a \cdot a^{-1} = a\left(\frac{1}{a}\right) = 1.$$

The operations of **subtraction** $(-)$ and **division** $(\div)$ are defined as follows.

Subtraction and Division

Definition	Meaning	Illustration
$a - b = a + (-b)$	To subtract one number from another, add the negative.	$3 - 7 = 3 + (-7)$
$a \div b = a \cdot \left(\dfrac{1}{b}\right)$ $= a \cdot b^{-1};\ b \neq 0$	To divide one number by a nonzero number, multiply by the reciprocal.	$3 \div 7 = 3 \cdot \left(\dfrac{1}{7}\right)$ $= 3 \cdot 7^{-1}$

We use either a/b or $\dfrac{a}{b}$ for $a \div b$ and refer to a/b as the **quotient of *a* and *b*** or the **fraction *a* over *b*.** The numbers a and b are the **numerator** and **denominator,** respectively, of a/b. Since 0 has no multiplicative inverse, a/b is not defined if $b = 0$; that is, *division by zero is not defined.* It is for this reason that the real numbers are not closed relative to division. Note that

$$1 \div b = \frac{1}{b} = b^{-1} \quad \text{if} \quad b \neq 0.$$

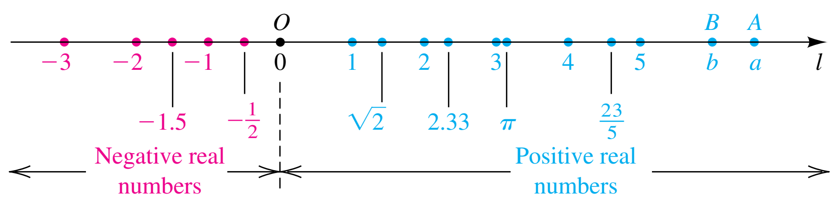
The following properties of quotients are true, provided all denominators are nonzero real numbers.

Properties of Quotients

Property	Illustration
(1) $\dfrac{a}{b} = \dfrac{c}{d}$ if $ad = bc$	$\dfrac{2}{5} = \dfrac{6}{15}$ because $2 \cdot 15 = 5 \cdot 6$
(2) $\dfrac{ad}{bd} = \dfrac{a}{b}$	$\dfrac{2 \cdot 3}{5 \cdot 3} = \dfrac{2}{5}$
(3) $\dfrac{a}{-b} = \dfrac{-a}{b} = -\dfrac{a}{b}$	$\dfrac{2}{-5} = \dfrac{-2}{5} = -\dfrac{2}{5}$
(4) $\dfrac{a}{b} + \dfrac{c}{b} = \dfrac{a + c}{b}$	$\dfrac{2}{5} + \dfrac{9}{5} = \dfrac{2 + 9}{5} = \dfrac{11}{5}$
(5) $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$	$\dfrac{2}{5} + \dfrac{4}{3} = \dfrac{2 \cdot 3 + 5 \cdot 4}{5 \cdot 3} = \dfrac{26}{15}$
(6) $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$	$\dfrac{2}{5} \cdot \dfrac{7}{3} = \dfrac{2 \cdot 7}{5 \cdot 3} = \dfrac{14}{15}$
(7) $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{ad}{bc}$	$\dfrac{2}{5} \div \dfrac{7}{3} = \dfrac{2}{5} \cdot \dfrac{3}{7} = \dfrac{6}{35}$

Real numbers may be represented by points on a line l such that to each real number a there corresponds exactly one point on l and to each point P on l there corresponds one real number. This is called a **one-to-one correspondence.** We first choose an arbitrary point O, called the **origin,** and associate with it the real number 0. Points associated with the integers are then determined by laying off successive line segments of equal length on either side of O, as illustrated in Figure 2. The point corresponding to a rational number, such as $\frac{23}{5}$, is obtained by subdividing these line segments. Points associated with certain irrational numbers, such as $\sqrt{2}$, can be found by construction (see Exercise 45).

Figure 2



The number a that is associated with a point A on l is the **coordinate** of A. We refer to these coordinates as a **coordinate system** and call l a **coordinate line** or a **real line.** A direction can be assigned to l by taking the **positive direction** to the right and the **negative direction** to the left. The positive direction is noted by placing an arrowhead on l, as shown in Figure 2.

The numbers that correspond to points to the right of O in Figure 2 are **positive real numbers.** Numbers that correspond to points to the left of O are **negative real numbers.** *The real number* 0 *is neither positive nor negative.*

Note the difference between a negative real number and the *negative of* a real number. In particular, the negative of a real number a can be positive. For example, if a is negative, say $a = -3$, then the negative of a is $-a = -(-3) = 3$, which is positive. In general, we have the following relationships.

Relationships Between a and $-a$

(1) If a is positive, then $-a$ is negative.

(2) If a is negative, then $-a$ is positive.

In the following chart we define the notions of **greater than** and **less than** for real numbers a and b. The symbols $>$ and $<$ are **inequality signs,** and the expressions $a > b$ and $a < b$ are called **(strict) inequalities.**

Greater Than or Less Than

Notation	Definition	Terminology
$a > b$	$a - b$ is positive	a is greater than b
$a < b$	$a - b$ is negative	a is less than b

If points A and B on a coordinate line have coordinates a and b, respectively, then $a > b$ is equivalent to the statement "A is to the *right* of B," whereas $a < b$ is equivalent to "A is to the *left* of B."

ILLUSTRATION **Greater Than (>) and Less Than (<)**

- $5 > 3$, since $5 - 3 = 2$ is positive.
- $-6 < -2$, since $-6 - (-2) = -6 + 2 = -4$ is negative.
- $\frac{1}{3} > 0.33$, since $\frac{1}{3} - 0.33 = \frac{1}{3} - \frac{33}{100} = \frac{1}{300}$ is positive.
- $7 > 0$, since $7 - 0 = 7$ is positive.
- $-4 < 0$, since $-4 - 0 = -4$ is negative.

The next law enables us to compare, or *order*, any two real numbers.

Trichotomy Law

If a and b are real numbers, then exactly one of the following is true:

$$a = b, \qquad a > b, \qquad \text{or} \qquad a < b$$

We refer to the **sign** of a real number as positive if the number is positive, or negative if the number is negative. Two real numbers have *the same sign* if both are positive or both are negative. The numbers have *opposite signs* if one is positive and the other is negative. The following results about the signs of products and quotients of two real numbers a and b can be proved using properties of negatives and quotients.

Laws of Signs	
	(1) If a and b have the same sign, then ab and $\dfrac{a}{b}$ are positive.
	(2) If a and b have opposite signs, then ab and $\dfrac{a}{b}$ are negative.

The **converses*** of the laws of signs are also true. For example, if a quotient is negative, then the numerator and denominator have opposite signs.

The notation $a \geq b$, read "a **is greater than or equal to** b," means that either $a > b$ or $a = b$ (but not both). For example, $a^2 \geq 0$ for every real number a. The symbol $a \leq b$, which is read "a **is less than or equal to** b," means that either $a < b$ or $a = b$. Expressions of the form $a \geq b$ and $a \leq b$ are called **nonstrict inequalities,** since a may be equal to b. As with the equality symbol, we may negate any inequality symbol by putting a slash through it—that is, $\not>$ means not greater than.

An expression of the form $a < b < c$ is called a **continued inequality** and means that both $a < b$ *and* $b < c$; we say "b **is between** a and c." Similarly, the expression $c > b > a$ means that both $c > b$ and $b > a$.

ILLUSTRATION **Ordering Three Real Numbers**

- $1 < 5 < \frac{11}{2}$
- $-4 < \frac{2}{3} < \sqrt{2}$
- $3 > -6 > -10$

There are other types of inequalities. For example, $a < b \leq c$ means both $a < b$ and $b \leq c$. Similarly, $a \leq b < c$ means both $a \leq b$ and $b < c$. Finally, $a \leq b \leq c$ means both $a \leq b$ and $b \leq c$.

EXAMPLE 2 **Determining the sign of a real number**

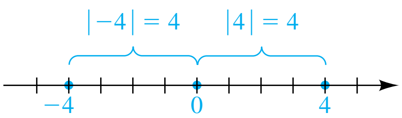
If $x > 0$ and $y < 0$, determine the sign of $\dfrac{x}{y} + \dfrac{y}{x}$.

SOLUTION Since x is a positive number and y is a negative number, x and y have opposite signs. Thus, both x/y and y/x are negative. The sum of two negative numbers is a negative number, so

$$\text{the sign of} \quad \frac{x}{y} + \frac{y}{x} \quad \text{is negative.}$$

*If a theorem is written in the form "if P, then Q," where P and Q are mathematical statements called the *hypothesis* and *conclusion,* respectively, then the *converse* of the theorem has the form "if Q, then P." If both the theorem and its converse are true, we often write "P if and only if Q" (denoted P iff Q).

Figure 3



If a is an integer, then it is the coordinate of some point A on a coordinate line, and the symbol $|a|$ denotes the number of units between A and the origin, without regard to direction. The nonnegative number $|a|$ is called the *absolute value of a*. Referring to Figure 3, we see that for the point with coordinate -4 we have $|-4| = 4$. Similarly, $|4| = 4$. In general, *if a is negative, we change its sign to find* $|a|$; *if a is nonnegative, then* $|a| = a$. The next definition extends this concept to every real number.

> **Definition of Absolute Value**
>
> The **absolute value** of a real number a, denoted by $|a|$, is defined as follows.
>
> **(1)** If $a \geq 0$, then $|a| = a$.
>
> **(2)** If $a < 0$, then $|a| = -a$.

Since a is negative in part (2) of the definition, $-a$ represents a *positive* real number. Some special cases of this definition are given in the following illustration.

ILLUSTRATION **The Absolute Value Notation $|a|$**

- $|3| = 3$, since $3 > 0$.
- $|-3| = -(-3)$, since $-3 < 0$. Thus, $|-3| = 3$.
- $|2 - \sqrt{2}| = 2 - \sqrt{2}$, since $2 - \sqrt{2} > 0$.
- $|\sqrt{2} - 2| = -(\sqrt{2} - 2)$, since $\sqrt{2} - 2 < 0$.
  Thus, $|\sqrt{2} - 2| = 2 - \sqrt{2}$.

In the preceding illustration, $|3| = |-3|$ and $|2 - \sqrt{2}| = |\sqrt{2} - 2|$. In general, we have the following:

$$|a| = |-a|, \text{ for every real number } a$$

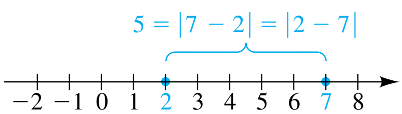
EXAMPLE 3 **Removing an absolute value symbol**

If $x < 1$, rewrite $|x - 1|$ without using the absolute value symbol.

SOLUTION If $x < 1$, then $x - 1 < 0$; that is, $x - 1$ is negative. Hence, by part (2) of the definition of absolute value,

$$|x - 1| = -(x - 1) = -x + 1 = 1 - x.$$

Figure 4



We shall use the concept of absolute value to define the distance between any two points on a coordinate line. First note that the distance between the points with coordinates 2 and 7, shown in Figure 4, equals 5 units. This distance is the difference obtained by subtracting the smaller (leftmost) coordinate from the larger (rightmost) coordinate ($7 - 2 = 5$). If we use absolute values, then, since $|7 - 2| = |2 - 7|$, it is unnecessary to be concerned about the order of subtraction. This fact motivates the next definition.

Definition of the Distance Between Points on a Coordinate Line

Let a and b be the coordinates of two points A and B, respectively, on a coordinate line. The **distance between A and B,** denoted by $d(A, B)$, is defined by

$$d(A, B) = |b - a|.$$

The number $d(A, B)$ is the length of the line segment AB.

Since $d(B, A) = |a - b|$ and $|b - a| = |a - b|$, we see that

$$d(A, B) = d(B, A).$$

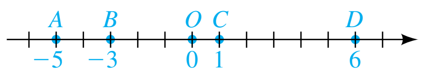
Note that the distance between the origin O and the point A is

$$d(O, A) = |a - 0| = |a|,$$

which agrees with the geometric interpretation of absolute value illustrated in Figure 4. The formula $d(A, B) = |b - a|$ is true regardless of the signs of a and b, as illustrated in the next example.

EXAMPLE 4 Finding distances between points

Figure 5



Let A, B, C, and D have coordinates -5, -3, 1, and 6, respectively, on a coordinate line, as shown in Figure 5. Find $d(A, B)$, $d(C, B)$, $d(O, A)$, and $d(C, D)$.

SOLUTION Using the definition of the distance between points on a coordinate line, we obtain the distances:

$$\begin{aligned} d(A, B) &= |-3 - (-5)| = |-3 + 5| = |2| = 2 \\ d(C, B) &= |-3 - 1| = |-4| = 4 \\ d(O, A) &= |-5 - 0| = |-5| = 5 \\ d(C, D) &= |6 - 1| = |5| = 5 \end{aligned}$$

The concept of absolute value has uses other than finding distances between points; it is employed whenever we are interested in the magnitude or numerical value of a real number without regard to its sign.

In the next section we shall discuss the *exponential notation* a^n, where a is a real number (called the *base*) and n is an integer (called an *exponent*). In particular, for base 10 we have

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 10 \cdot 10 = 100, \quad 10^3 = 10 \cdot 10 \cdot 10 = 1000,$$

and so on. For negative exponents we use the reciprocal of the corresponding positive exponent, as follows:

$$10^{-1} = \frac{1}{10^1} = \frac{1}{10}, \quad 10^{-2} = \frac{1}{10^2} = \frac{1}{100}, \quad 10^{-3} = \frac{1}{10^3} = \frac{1}{1000}$$

We can use this notation to write any finite decimal representation of a real number as a sum of the following type:

$$\begin{aligned} 437.56 &= 4(100) + 3(10) + 7(1) + 5\left(\tfrac{1}{10}\right) + 6\left(\tfrac{1}{100}\right) \\ &= 4(10^2) + 3(10^1) + 7(10^0) + 5(10^{-1}) + 6(10^{-2}) \end{aligned}$$

In the sciences it is often necessary to work with very large or very small numbers and to compare the relative magnitudes of very large or very small quantities. We usually represent a large or small positive number a in *scientific form,* using the symbol $\times$ to denote multiplication.

Scientific Form	$a = c \times 10^n$, where $1 \le c < 10$ and n is an integer

The distance a ray of light travels in one year is approximately 5,900,000,000,000 miles. This number may be written in scientific form as 5.9×10^{12}. The positive exponent 12 indicates that the decimal point should be moved 12 places to the *right.* The notation works equally well for small numbers. The weight of an oxygen molecule is estimated to be

$$0.000\ 000\ 000\ 000\ 000\ 000\ 000\ 053 \text{ gram,}$$

or, in scientific form, 5.3×10^{-23} gram. The negative exponent indicates that the decimal point should be moved 23 places to the *left*.

ILLUSTRATION **Scientific Form**

- $513 = 5.13 \times 10^2$
- $93{,}000{,}000 = 9.3 \times 10^7$
- $0.000\ 000\ 000\ 43 = 4.3 \times 10^{-10}$
- $7.3 = 7.3 \times 10^0$
- $20{,}700 = 2.07 \times 10^4$
- $0.000\ 648 = 6.48 \times 10^{-4}$

Figure 6

2.025 13

or

2.025E13

or

2.025 ^{13}

Many calculators use scientific form in their display panels. For the number $c \times 10^n$, the 10 is suppressed and the exponent is often shown preceded by the letter E. For example, to find $(4{,}500{,}000)^2$ on a scientific calculator, we could enter the integer 4,500,000 and press the $\boxed{x^2}$ (or squaring) key, obtaining a display similar to one of those in Figure 6. We would translate this as 2.025×10^{13}. Thus,

$$(4{,}500{,}000)^2 = 20{,}250{,}000{,}000{,}000.$$

Calculators may also use scientific form in the entry of numbers. The user's manual for your calculator should give specific details.

Before we conclude this section, we should briefly consider the issue of rounding off results. Applied problems often include numbers that are obtained by various types of measurements and, hence, are *approximations* to exact values. Such answers should be rounded off, since the final result of a calculation cannot be more accurate than the data that have been used. For example, if the length and width of a rectangle are measured to two-decimal-place accuracy, we cannot expect more than two-decimal-place accuracy in the calculated value of the area of the rectangle. For purely *mathematical* work, if values of the length and width of a rectangle are given, we assume that the dimensions are *exact,* and no rounding off is required.

If a number a is written in scientific form as $a = c \times 10^n$ for $1 \le c < 10$ and if c is rounded off to k decimal places, then we say that a is accurate (or has been rounded off) to $k + 1$ **significant figures,** or **digits.** For example, 37.2638 rounded to 5 significant figures is 3.7264×10^1, or 37.264; to 3 significant figures, 3.73×10^1, or 37.3; and to 1 significant figure, 4×10^1, or 40.

1.1 *Exercises*

Exer. 1–2: If $x < 0$ and $y > 0$, determine the sign of the real number.

1 (a) xy (b) x^2y (c) $\dfrac{x}{y} + x$ (d) $y - x$

2 (a) $\dfrac{x}{y}$ (b) xy^2 (c) $\dfrac{x - y}{xy}$ (d) $y(y - x)$

Exer. 3–6: Replace the symbol $\square$ with either <, >, or = to make the resulting statement true.

3 (a) $-7 \square -4$ (b) $\dfrac{\pi}{2} \square 1.57$ (c) $\sqrt{225} \square 15$

4 (a) $-3 \square -5$ (b) $\dfrac{\pi}{4} \square 0.8$ (c) $\sqrt{289} \square 17$

5 (a) $\frac{1}{11} \square 0.09$ (b) $\frac{2}{3} \square 0.6666$ (c) $\frac{22}{7} \square \pi$

6 (a) $\frac{1}{7} \square 0.143$ (b) $\frac{5}{6} \square 0.833$ (c) $\sqrt{2} \square 1.4$

Exer. 7–8: Express the statement as an inequality.

7 (a) x is negative.

(b) y is nonnegative.

(c) q is less than or equal to π.

(d) d is between 4 and 2.

(e) t is not less than 5.

(f) The negative of z is not greater than 3.

(g) The quotient of p and q is at most 7.

(h) The reciprocal of w is at least 9.

(i) The absolute value of x is greater than 7.

8 (a) b is positive.

(b) s is nonpositive.

(c) w is greater than or equal to -4.

(d) c is between $\frac{1}{5}$ and $\frac{1}{3}$.

(e) p is not greater than -2.

(f) The negative of m is not less than -2.

(g) The quotient of r and s is at least $\frac{1}{5}$.

(h) The reciprocal of f is at most 14.

(i) The absolute value of x is less than 4.

Exer. 9–14: Rewrite the number without using the absolute value symbol, and simplify the result.

9 (a) $|-3 - 2|$ (b) $|-5| - |2|$ (c) $|7| + |-4|$

10 (a) $|-11 + 1|$ (b) $|6| - |-3|$ (c) $|8| + |-9|$

11 (a) $(-5)|3 - 6|$ (b) $|-6|/(-2)$ (c) $|-7| + |4|$

12 (a) $(4)|6 - 7|$ (b) $5/|-2|$ (c) $|-1| + |-9|$

13 (a) $|4 - \pi|$ (b) $|\pi - 4|$ (c) $|\sqrt{2} - 1.5|$

14 (a) $|\sqrt{3} - 1.7|$ (b) $|1.7 - \sqrt{3}|$ (c) $|\frac{1}{5} - \frac{1}{3}|$

Exer. 15–18: The given numbers are coordinates of points A, B, and C, respectively, on a coordinate line. Find the distance.

(a) $d(A, B)$ **(b) $d(B, C)$**

(c) $d(C, B)$ **(d) $d(A, C)$**

15 3, 7, -5 16 -6, -2, 4

17 -9, 1, 10 18 8, -4, -1

Exer. 19–24: The two given numbers are coordinates of points A and B, respectively, on a coordinate line. Express the indicated statement as an inequality involving the absolute value symbol.

19 x, 7; $d(A, B)$ is less than 5

20 x, $-\sqrt{2}$; $d(A, B)$ is greater than 1

21 x, -3; $d(A, B)$ is at least 8

22 x, 4; $d(A, B)$ is at most 2

23 4, x; $d(A, B)$ is not greater than 3

24 -2, x; $d(A, B)$ is not less than 2

Exer. 25–32: Rewrite the expression without using the absolute value symbol, and simplify the result.

25 $|3 + x|$ if $x < -3$ 26 $|5 - x|$ if $x > 5$

27 $|2 - x|$ if $x < 2$ 28 $|7 + x|$ if $x \geq -7$

29 $|a - b|$ if $a < b$ 30 $|a - b|$ if $a > b$

31 $|x^2 + 4|$ 32 $|-x^2 - 1|$

Exer. 33–40: Replace the symbol $\square$ with either $=$ or $\neq$ to make the resulting statement true for all real numbers a, b, c, and d, whenever the expressions are defined.

33 $\dfrac{ab + ac}{a} \square b + ac$ 34 $\dfrac{ab + ac}{a} \square b + c$

35 $\dfrac{b + c}{a} \square \dfrac{b}{a} + \dfrac{c}{a}$ 36 $\dfrac{a + c}{b + d} \square \dfrac{a}{b} + \dfrac{c}{d}$

37 $(a \div b) \div c \square a \div (b \div c)$

38 $(a - b) - c \square a - (b - c)$

39 $\dfrac{a - b}{b - a} \square -1$ 40 $-(a + b) \square -a + b$

Exer. 41–42: Approximate the real-number expression to four decimal places.

41 (a) $|3.2^2 - \sqrt{3.15}|$

(b) $\sqrt{(15.6 - 1.5)^2 + (4.3 - 5.4)^2}$

42 (a) $\dfrac{3.42 - 1.29}{5.83 + 2.64}$

(b) π^3

Exer. 43–44: Approximate the real-number expression. Express the answer in scientific notation accurate to four significant figures.

43 (a) $\dfrac{1.2 \times 10^3}{3.1 \times 10^2 + 1.52 \times 10^3}$

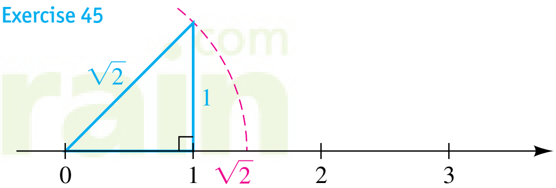
(b) $(1.23 \times 10^{-4}) + \sqrt{4.5 \times 10^3}$

44 (a) $\sqrt{|3.45 - 1.2 \times 10^4| + 10^5}$

(b) $(1.791 \times 10^2) \times (9.84 \times 10^3)$

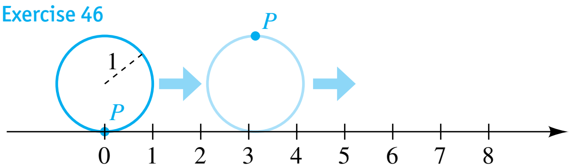
45 The point on a coordinate line corresponding to $\sqrt{2}$ may be determined by constructing a right triangle with sides of length 1, as shown in the figure. Determine the points that correspond to $\sqrt{3}$ and $\sqrt{5}$, respectively. (*Hint:* Use the Pythagorean theorem.)

Exercise 45



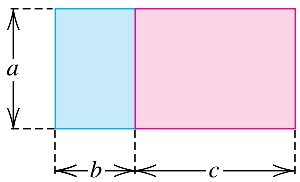
46 A circle of radius 1 rolls along a coordinate line in the positive direction, as shown in the figure. If point P is initially at the origin, find the coordinate of P after one, two, and ten complete revolutions.

Exercise 46



47 Geometric proofs of properties of real numbers were first given by the ancient Greeks. In order to establish the distributive property $a(b + c) = ab + ac$ for positive real numbers a, b, and c, find the area of the rectangle shown in the figure on the next page in two ways.

Exercise 47



48 Rational approximations to square roots can be found using a formula discovered by the ancient Babylonians. Let x_1 be the first rational approximation for $\sqrt{n}$. If we let

$$x_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right),$$

then x_2 will be a better approximation for $\sqrt{n}$, and we can repeat the computation with x_2 replacing x_1. Starting with $x_1 = \frac{3}{2}$, find the next two rational approximations for $\sqrt{2}$.

Exer. 49–50: Express the number in scientific form.

49 (a) 427,000 (b) 0.000 000 098 (c) 810,000,000

50 (a) 85,200 (b) 0.000 005 5 (c) 24,900,000

Exer. 51–52: Express the number in decimal form.

51 (a) 8.3×10^5 (b) 2.9×10^{-12} (c) 5.63×10^8

52 (a) 2.3×10^7 (b) 7.01×10^{-9} (c) 1.23×10^{10}

53 **Mass of a hydrogen atom** The mass of a hydrogen atom is approximately

0.000 000 000 000 000 000 000 001 7 gram.

Express this number in scientific form.

54 **Mass of an electron** The mass of an electron is approximately 9.1×10^{-31} kilogram. Express this number in decimal form.

55 **Light year** In astronomy, distances to stars are measured in light years. One light year is the distance a ray of light travels in one year. If the speed of light is approximately 186,000 miles per second, estimate the number of miles in one light year.

56 **Milky Way galaxy**

(a) Astronomers have estimated that the Milky Way galaxy contains 100 billion stars. Express this number in scientific form.

(b) The diameter d of the Milky Way galaxy is estimated as 100,000 light years. Express d in miles. (Refer to Exercise 55.)

57 **Avogadro's number** The number of hydrogen atoms in a mole is Avogadro's number, 6.02×10^{23}. If one mole of the gas has a mass of 1.01 grams, estimate the mass of a hydrogen atom.

58 **Fish population** The population dynamics of many fish are characterized by extremely high fertility rates among adults and very low survival rates among the young. A mature halibut may lay as many as 2.5 million eggs, but only 0.00035% of the offspring survive to the age of 3 years. Use scientific form to approximate the number of offspring that live to age 3.

59 **Frames in a movie film** One of the longest movies ever made is a 1970 British film that runs for 48 hours. Assuming that the film speed is 24 frames per second, approximate the total number of frames in this film. Express your answer in scientific form.

60 **Large prime numbers** The number $2^{44,497} - 1$ is prime. At the time that this number was determined to be prime, it took one of the world's fastest computers about 60 days to verify that it was prime. This computer was capable of performing 2×10^{11} calculations per second. Use scientific form to estimate the number of calculations needed to perform this computation. (More recently, in 2005, $2^{30,402,457} - 1$, a number containing 9,152,052 digits, was shown to be prime.)

61 **Tornado pressure** When a tornado passes near a building, there is a rapid drop in the outdoor pressure and the indoor pressure does not have time to change. The resulting difference is capable of causing an outward pressure of 1.4 lb/in^2 on the walls and ceiling of the building.

(a) Calculate the force in pounds exerted on 1 square foot of a wall.

(b) Estimate the tons of force exerted on a wall that is 8 feet high and 40 feet wide.

62 **Cattle population** A rancher has 750 head of cattle consisting of 400 adults (aged 2 or more years), 150 yearlings, and 200 calves. The following information is known about this particular species. Each spring an adult female gives birth to a single calf, and 75% of these calves will survive the first year. The yearly survival percentages for yearlings and adults are 80% and 90%, respectively. The male-female ratio is one in all age classes. Estimate the population of each age class

(a) next spring (b) last spring

1.2 Exponents and Radicals

If n is a positive integer, the exponential notation a^n, defined in the following chart, represents the product of the real number a with itself n times. We refer to a^n as **a to the nth power** or, simply, *a to the n*. The positive integer n is called the **exponent,** and the real number a is called the **base.**

Exponential Notation

General case (n is any positive integer)	Special cases
$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}$	$a^1 = a$ $a^2 = a \cdot a$ $a^3 = a \cdot a \cdot a$ $a^6 = a \cdot a \cdot a \cdot a \cdot a \cdot a$

The next illustration contains several numerical examples of exponential notation.

ILLUSTRATION **The Exponential Notation a^n**

- $5^4 = 5 \cdot 5 \cdot 5 \cdot 5 = 625$
- $\left(\frac{1}{2}\right)^5 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32}$
- $(-3)^3 = (-3)(-3)(-3) = -27$
- $\left(-\frac{1}{3}\right)^4 = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) = \left(\frac{1}{9}\right)\left(\frac{1}{9}\right) = \frac{1}{81}$

It is important to note that if n is a positive integer, then an expression such as $3a^n$ means $3(a^n)$, *not* $(3a)^n$. The real number 3 is the **coefficient** of a^n in the expression $3a^n$. Similarly, $-3a^n$ means $(-3)a^n$, *not* $(-3a)^n$.

ILLUSTRATION **The Notation ca^n**

- $5 \cdot 2^3 = 5 \cdot 8 = 40$
- $-5 \cdot 2^3 = -5 \cdot 8 = -40$
- $-2^4 = -(2^4) = -16$
- $3(-2)^3 = 3(-2)(-2)(-2) = 3(-8) = -24$

We next extend the definition of a^n to nonpositive exponents.

Zero and Negative (Nonpositive) Exponents

Definition ($a \neq 0$)	Illustrations	
$a^0 = 1$	$3^0 = 1,$	$\left(-\sqrt{2}\right)^0 = 1$
$a^{-n} = \dfrac{1}{a^n}$	$5^{-3} = \dfrac{1}{5^3},$	$(-3)^{-5} = \dfrac{1}{(-3)^5}$

If m and n are positive integers, then

$$a^m a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors of } a}.$$

Since the total number of factors of a on the right is $m + n$, this expression is equal to a^{m+n}; that is,

$$a^m a^n = a^{m+n}.$$

We can extend this formula to $m \le 0$ or $n \le 0$ by using the definitions of the zero exponent and negative exponents. This gives us law 1, stated in the next chart.

To prove law 2, we may write, for m and n positive,

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot a^m \cdot \cdots \cdot a^m}_{n \text{ factors of } a^m}$$

and count the number of times a appears as a factor on the right-hand side. Since $a^m = a \cdot a \cdot a \cdot \cdots \cdot a$, with a occurring as a factor m times, and since the number of such groups of m factors is n, the total number of factors of a is $m \cdot n$. Thus,

$$(a^m)^n = a^{mn}.$$

The cases $m \le 0$ and $n \le 0$ can be proved using the definition of nonpositive exponents. The remaining three laws can be established in similar fashion by counting factors. In laws 4 and 5 we assume that denominators are not 0.

Laws of Exponents for Real Numbers a and b and Integers m and n

Law	Illustration
(1) $a^m a^n = a^{m+n}$	$2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$
(2) $(a^m)^n = a^{mn}$	$(2^3)^4 = 2^{3\cdot 4} = 2^{12} = 4096$
(3) $(ab)^n = a^n b^n$	$(20)^3 = (2 \cdot 10)^3 = 2^3 \cdot 10^3 = 8 \cdot 1000 = 8000$
(4) $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	$\left(\dfrac{2}{5}\right)^3 = \dfrac{2^3}{5^3} = \dfrac{8}{125}$
(5) (a) $\dfrac{a^m}{a^n} = a^{m-n}$	$\dfrac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$
(b) $\dfrac{a^m}{a^n} = \dfrac{1}{a^{n-m}}$	$\dfrac{2^3}{2^5} = \dfrac{1}{2^{5-3}} = \dfrac{1}{2^2} = \dfrac{1}{4}$

We usually use 5(a) if $m > n$ and 5(b) if $m < n$.

We can extend laws of exponents to obtain rules such as $(abc)^n = a^n b^n c^n$ and $a^m a^n a^p = a^{m+n+p}$. Some other examples of the laws of exponents are given in the next illustration.

ILLUSTRATION **Laws of Exponents**

- $x^5x^6x^2 = x^{5+6+2} = x^{13}$
- $(y^5)^7 = y^{5\cdot 7} = y^{35}$
- $(3st)^4 = 3^4s^4t^4 = 81s^4t^4$
- $\left(\dfrac{p}{2}\right)^5 = \dfrac{p^5}{2^5} = \dfrac{p^5}{32}$
- $\dfrac{c^8}{c^3} = c^{8-3} = c^5$
- $\dfrac{u^3}{u^8} = \dfrac{1}{u^{8-3}} = \dfrac{1}{u^5}$

To **simplify** an expression involving powers of real numbers means to change it to an expression in which each real number appears only once and all exponents are positive. *We shall assume that denominators always represent nonzero real numbers.*

EXAMPLE 1 **Simplifying expressions containing exponents**

Use laws of exponents to simplify each expression:

(a) $(3x^3y^4)(4xy^5)$ **(b)** $(2a^2b^3c)^4$ **(c)** $\left(\dfrac{2r^3}{s}\right)^2\left(\dfrac{s}{r^3}\right)^3$ **(d)** $(u^{-2}v^3)^{-3}$

SOLUTION

(a)
$$\begin{aligned}(3x^3y^4)(4xy^5) &= (3)(4)x^3xy^4y^5 && \text{rearrange factors}\\ &= 12x^4y^9 && \text{law 1}\end{aligned}$$

(b)
$$\begin{aligned}(2a^2b^3c)^4 &= 2^4(a^2)^4(b^3)^4c^4 && \text{law 3}\\ &= 16a^8b^{12}c^4 && \text{law 2}\end{aligned}$$

(c)
$$\begin{aligned}\left(\frac{2r^3}{s}\right)^2\left(\frac{s}{r^3}\right)^3 &= \frac{(2r^3)^2}{s^2}\cdot\frac{s^3}{(r^3)^3} && \text{law 4}\\ &= \frac{2^2(r^3)^2}{s^2}\cdot\frac{s^3}{(r^3)^3} && \text{law 3}\\ &= \left(\frac{4r^6}{s^2}\right)\left(\frac{s^3}{r^9}\right) && \text{law 2}\\ &= 4\left(\frac{r^6}{r^9}\right)\left(\frac{s^3}{s^2}\right) && \text{rearrange factors}\\ &= 4\left(\frac{1}{r^3}\right)(s) && \text{laws 5(b) and 5(a)}\\ &= \frac{4s}{r^3} && \text{rearrange factors}\end{aligned}$$

(d)
$$\begin{aligned}(u^{-2}v^3)^{-3} &= (u^{-2})^{-3}(v^3)^{-3} && \text{law 3}\\ &= u^6v^{-9} && \text{law 2}\\ &= \frac{u^6}{v^9} && \text{definition of } a^{-n}\end{aligned}$$

The following theorem is useful for problems that involve negative exponents.

Theorem on Negative Exponents	**(1)** $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$ **(2)** $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

PROOFS Using properties of negative exponents and quotients, we obtain

(1) $\dfrac{a^{-m}}{b^{-n}} = \dfrac{1/a^m}{1/b^n} = \dfrac{1}{a^m} \cdot \dfrac{b^n}{1} = \dfrac{b^n}{a^m}$

(2) $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{a^{-n}}{b^{-n}} = \dfrac{b^n}{a^n} = \left(\dfrac{b}{a}\right)^n$

EXAMPLE 2 **Simplifying expressions containing negative exponents**

Simplify:

(a) $\dfrac{8x^3y^{-5}}{4x^{-1}y^2}$ **(b)** $\left(\dfrac{u^2}{2v}\right)^{-3}$

SOLUTION We apply the theorem on negative exponents and the laws of exponents.

(a) $\dfrac{8x^3y^{-5}}{4x^{-1}y^2} = \dfrac{8x^3}{4y^2} \cdot \dfrac{y^{-5}}{x^{-1}}$ rearrange quotients so that negative exponents are in one fraction

$= \dfrac{8x^3}{4y^2} \cdot \dfrac{x^1}{y^5}$ theorem on negative exponents (1)

$= \dfrac{2x^4}{y^7}$ law 1 of exponents

(b) $\left(\dfrac{u^2}{2v}\right)^{-3} = \left(\dfrac{2v}{u^2}\right)^3$ theorem on negative exponents (2)

$= \dfrac{2^3v^3}{(u^2)^3}$ laws 4 and 3 of exponents

$= \dfrac{8v^3}{u^6}$ law 2 of exponents

We next define the **principal *n*th root** $\sqrt[n]{a}$ of a real number a.

Definition of $\sqrt[n]{a}$

Let n be a positive integer greater than 1, and let a be a real number.

(1) If $a = 0$, then $\sqrt[n]{a} = 0$.

(2) If $a > 0$, then $\sqrt[n]{a}$ is the *positive* real number b such that $b^n = a$.

(3) **(a)** If $a < 0$ and n is odd, then $\sqrt[n]{a}$ is the *negative* real number b such that $b^n = a$.

(b) If $a < 0$ and n is even, then $\sqrt[n]{a}$ is not a real number.

Complex numbers, discussed in Section 2.4, are needed to define $\sqrt[n]{a}$ if $a < 0$ and n is an *even* positive integer, because for all real numbers b, $b^n \geq 0$ whenever n is even.

If $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$ and call $\sqrt{a}$ the **principal square root** of a or, simply, the **square root** of a. The number $\sqrt[3]{a}$ is the (principal) **cube root** of a.

ILLUSTRATION **The Principal nth Root $\sqrt[n]{a}$**

- $\sqrt{16} = 4$, since $4^2 = 16$.
- $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$, since $\left(\frac{1}{2}\right)^5 = \frac{1}{32}$.
- $\sqrt[3]{-8} = -2$, since $(-2)^3 = -8$.
- $\sqrt[4]{-16}$ is not a real number.

Note that $\sqrt{16} \neq \pm 4$, since, by definition, roots of positive real numbers are positive. The symbol $\pm$ is read "plus or minus."

To complete our terminology, the expression $\sqrt[n]{a}$ is a **radical,** the number a is the **radicand,** and n is the **index** of the radical. The symbol $\sqrt{\ }$ is called a **radical sign.**

If $\sqrt{a} = b$, then $b^2 = a$; that is, $\left(\sqrt{a}\right)^2 = a$. If $\sqrt[3]{a} = b$, then $b^3 = a$, or $\left(\sqrt[3]{a}\right)^3 = a$. Generalizing this pattern gives us property 1 in the next chart.

Properties of $\sqrt[n]{a}$ (n is a positive integer)

Property	Illustrations	
(1) $\left(\sqrt[n]{a}\right)^n = a$ if $\sqrt[n]{a}$ is a real number	$\left(\sqrt{5}\right)^2 = 5$,	$\left(\sqrt[3]{-8}\right)^3 = -8$
(2) $\sqrt[n]{a^n} = a$ if $a \geq 0$	$\sqrt{5^2} = 5$,	$\sqrt[3]{2^3} = 2$
(3) $\sqrt[n]{a^n} = a$ if $a < 0$ and n is odd	$\sqrt[3]{(-2)^3} = -2$,	$\sqrt[5]{(-2)^5} = -2$
(4) $\sqrt[n]{a^n} = \lvert a \rvert$ if $a < 0$ and n is even	$\sqrt{(-3)^2} = \lvert -3 \rvert = 3$,	$\sqrt[4]{(-2)^4} = \lvert -2 \rvert = 2$

If $a \geq 0$, then property 4 reduces to property 2. We also see from property 4 that

$$\sqrt{x^2} = \lvert x \rvert$$

for every real number x. In particular, if $x \geq 0$, then $\sqrt{x^2} = x$; however, if $x < 0$, then $\sqrt{x^2} = -x$, which is positive.

The three laws listed in the next chart are true for positive integers m and n, *provided the indicated roots* ***exist***—that is, provided the roots are real numbers.

Laws of Radicals

Law	Illustrations
(1) $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$	$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\,\sqrt{2} = 5\sqrt{2}$ $\sqrt[3]{-108} = \sqrt[3]{(-27)(4)} = \sqrt[3]{-27}\,\sqrt[3]{4} = -3\sqrt[3]{4}$
(2) $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[3]{\dfrac{5}{8}} = \dfrac{\sqrt[3]{5}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{5}}{2}$
(3) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt{\sqrt[3]{64}} = \sqrt[2(3)]{64} = \sqrt[6]{2^6} = 2$

The radicands in laws 1 and 2 involve products and quotients. Care must be taken if sums or differences occur in the radicand. The following chart contains two particular warnings concerning commonly made mistakes.

Warning!

If $a \neq 0$ and $b \neq 0$	Illustration
(1) $\sqrt{a^2 + b^2} \neq a + b$	$\sqrt{3^2 + 4^2} = \sqrt{25} = 5 \neq 3 + 4 = 7$
(2) $\sqrt{a + b} \neq \sqrt{a} + \sqrt{b}$	$\sqrt{4 + 9} = \sqrt{13} \neq \sqrt{4} + \sqrt{9} = 5$

If c is a real number and c^n occurs as a factor in a radical of index n, then we can remove c from the radicand if the sign of c is taken into account. For example, if $c > 0$ or if $c < 0$ and n is *odd,* then

$$\sqrt[n]{c^n d} = \sqrt[n]{c^n}\,\sqrt[n]{d} = c\sqrt[n]{d},$$

provided $\sqrt[n]{d}$ exists. If $c < 0$ and n is *even,* then

$$\sqrt[n]{c^n d} = \sqrt[n]{c^n}\,\sqrt[n]{d} = |c|\sqrt[n]{d},$$

provided $\sqrt[n]{d}$ exists.

ILLUSTRATION Removing nth Powers from $\sqrt[n]{}$

- $\sqrt[5]{x^7} = \sqrt[5]{x^5 \cdot x^2} = \sqrt[5]{x^5}\,\sqrt[5]{x^2} = x\sqrt[5]{x^2}$
- $\sqrt[3]{x^7} = \sqrt[3]{x^6 \cdot x} = \sqrt[3]{(x^2)^3 x} = \sqrt[3]{(x^2)^3}\,\sqrt[3]{x} = x^2\sqrt[3]{x}$
- $\sqrt{x^2 y} = \sqrt{x^2}\,\sqrt{y} = |x|\sqrt{y}$
- $\sqrt{x^6} = \sqrt{(x^3)^2} = |x^3|$
- $\sqrt[4]{x^6 y^3} = \sqrt[4]{x^4 \cdot x^2 y^3} = \sqrt[4]{x^4}\,\sqrt[4]{x^2 y^3} = |x|\sqrt[4]{x^2 y^3}$

Note: To avoid considering absolute values, *in examples and exercises involving radicals in this chapter, we shall assume that all letters—a, b, c, d, x, y,* *and so on—that appear in radicands represent positive real numbers, unless otherwise specified.*

As shown in the preceding illustration and in the following examples, if the index of a radical is n, then we rearrange the radicand, isolating a factor of the form p^n, where p may consist of several letters. We then remove $\sqrt[n]{p^n} = p$ from the radical, as previously indicated. Thus, in Example 3(b) the index of the radical is 3 and we rearrange the radicand into *cubes,* obtaining a factor p^3, with $p = 2xy^2z$. In part (c) the index of the radical is 2 and we rearrange the radicand into squares, obtaining a factor p^2, with $p = 3a^3b^2$.

To *simplify a radical* means to remove factors from the radical until no factor in the radicand has an exponent greater than or equal to the index of the radical and the index is as low as possible.

EXAMPLE 3 Removing factors from radicals

Simplify each radical (all letters denote positive real numbers):

(a) $\sqrt[3]{320}$ **(b)** $\sqrt[3]{16x^3y^8z^4}$ **(c)** $\sqrt{3a^2b^3}\,\sqrt{6a^5b}$

SOLUTION

(a)

$$\begin{aligned} \sqrt[3]{320} &= \sqrt[3]{64 \cdot 5} && \text{factor out the largest cube in 320} \\ &= \sqrt[3]{4^3}\,\sqrt[3]{5} && \text{law 1 of radicals} \\ &= 4\sqrt[3]{5} && \text{property 2 of } \sqrt[n]{\ } \end{aligned}$$

(b)

$$\begin{aligned} \sqrt[3]{16x^3y^8z^4} &= \sqrt[3]{(2^3x^3y^6z^3)(2y^2z)} && \text{rearrange radicand into cubes} \\ &= \sqrt[3]{(2xy^2z)^3(2y^2z)} && \text{laws 2 and 3 of exponents} \\ &= \sqrt[3]{(2xy^2z)^3}\,\sqrt[3]{2y^2z} && \text{law 1 of radicals} \\ &= 2xy^2z\sqrt[3]{2y^2z} && \text{property 2 of } \sqrt[n]{\ } \end{aligned}$$

(c)

$$\begin{aligned} \sqrt{3a^2b^3}\,\sqrt{6a^5b} &= \sqrt{3a^2b^3 \cdot 2 \cdot 3a^5b} && \text{law 1 of radicals} \\ &= \sqrt{(3^2a^6b^4)(2a)} && \text{rearrange radicand into squares} \\ &= \sqrt{(3a^3b^2)^2(2a)} && \text{laws 2 and 3 of exponents} \\ &= \sqrt{(3a^3b^2)^2}\,\sqrt{2a} && \text{law 1 of radicals} \\ &= 3a^3b^2\sqrt{2a} && \text{property 2 of } \sqrt[n]{\ } \end{aligned}$$

If the denominator of a quotient contains a factor of the form $\sqrt[n]{a^k}$, with $k < n$ and $a > 0$, then multiplying the numerator and denominator by $\sqrt[n]{a^{n-k}}$ will eliminate the radical from the denominator, since

$$\sqrt[n]{a^k}\,\sqrt[n]{a^{n-k}} = \sqrt[n]{a^{k+n-k}} = \sqrt[n]{a^n} = a.$$

This process is called **rationalizing a denominator.** Some special cases are listed in the following chart.

Rationalizing Denominators of Quotients ($a > 0$)

Factor in denominator	Multiply numerator and denominator by	Resulting factor
$\sqrt{a}$	$\sqrt{a}$	$\sqrt{a}\,\sqrt{a} = \sqrt{a^2} = a$
$\sqrt[3]{a}$	$\sqrt[3]{a^2}$	$\sqrt[3]{a}\,\sqrt[3]{a^2} = \sqrt[3]{a^3} = a$
$\sqrt[7]{a^3}$	$\sqrt[7]{a^4}$	$\sqrt[7]{a^3}\,\sqrt[7]{a^4} = \sqrt[7]{a^7} = a$

The next example illustrates this technique.

EXAMPLE 4 **Rationalizing denominators**

Rationalize each denominator:

(a) $\dfrac{1}{\sqrt{5}}$ (b) $\dfrac{1}{\sqrt[3]{x}}$ (c) $\sqrt{\dfrac{2}{3}}$ (d) $\sqrt[5]{\dfrac{x}{y^2}}$

SOLUTION

(a) $$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5^2}} = \frac{\sqrt{5}}{5}$$

(b) $$\frac{1}{\sqrt[3]{x}} = \frac{1}{\sqrt[3]{x}}\frac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \frac{\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \frac{\sqrt[3]{x^2}}{x}$$

(c) $$\sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{2 \cdot 3}}{\sqrt{3^2}} = \frac{\sqrt{6}}{3}$$

(d) $$\sqrt[5]{\frac{x}{y^2}} = \frac{\sqrt[5]{x}}{\sqrt[5]{y^2}} = \frac{\sqrt[5]{x}}{\sqrt[5]{y^2}}\frac{\sqrt[5]{y^3}}{\sqrt[5]{y^3}} = \frac{\sqrt[5]{xy^3}}{\sqrt[5]{y^5}} = \frac{\sqrt[5]{xy^3}}{y}$$

If we use a calculator to find decimal approximations of radicals, there is no advantage in rationalizing denominators, such as $1/\sqrt{5} = \sqrt{5}/5$ or $\sqrt{2/3} = \sqrt{6}/3$, as we did in Example 4(a) and (c). However, for *algebraic* simplifications, changing expressions to such forms is sometimes desirable. Similarly, in advanced mathematics courses such as calculus, changing $1/\sqrt[3]{x}$ to $\sqrt[3]{x^2}/x$, as in Example 4(b), could make a problem *more* complicated. In such courses it is simpler to work with the expression $1/\sqrt[3]{x}$ than with its rationalized form.

We next use radicals to define *rational exponents*.

Definition of Rational Exponents

Let m/n be a rational number, where n is a positive integer greater than 1. If a is a real number such that $\sqrt[n]{a}$ exists, then

(1) $a^{1/n} = \sqrt[n]{a}$

(2) $a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

(3) $a^{m/n} = (a^{1/n})^m = (a^m)^{1/n}$

When evaluating $a^{m/n}$ in (2), we usually use $(\sqrt[n]{a})^m$; that is, we take the nth root of a first and then raise that result to the mth power, as shown in the following illustration.

ILLUSTRATION **The Exponential Notation $a^{m/n}$**

- $x^{1/3} = \sqrt[3]{x}$
- $x^{3/5} = (\sqrt[5]{x})^3 = \sqrt[5]{x^3}$
- $125^{2/3} = (\sqrt[3]{125})^2 = (\sqrt[3]{5^3})^2 = 5^2 = 25$
- $\left(\frac{32}{243}\right)^{3/5} = \left(\sqrt[5]{\frac{32}{243}}\right)^3 = \left(\sqrt[5]{\left(\frac{2}{3}\right)^5}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

The laws of exponents are true for rational exponents and also for *irrational* exponents, such as $3^{\sqrt{2}}$ or 5^{π}, considered in Chapter 5.

To simplify an expression involving rational powers of letters that represent real numbers, we change it to an expression in which each letter appears only once and all exponents are positive. As we did with radicals, we shall assume that all letters represent positive real numbers unless otherwise specified.

EXAMPLE 5 **Simplifying rational powers**

Simplify:

(a) $(-27)^{2/3}(4)^{-5/2}$ **(b)** $(r^2s^6)^{1/3}$ **(c)** $\left(\dfrac{2x^{2/3}}{y^{1/2}}\right)^2\left(\dfrac{3x^{-5/6}}{y^{1/3}}\right)$

SOLUTION

(a)

$$\begin{aligned}
(-27)^{2/3}(4)^{-5/2} &= (\sqrt[3]{-27})^2(\sqrt{4})^{-5} && \text{definition of rational exponents}\\
&= (-3)^2(2)^{-5} && \text{take roots}\\
&= \frac{(-3)^2}{2^5} && \text{definition of negative exponents}\\
&= \frac{9}{32} && \text{take powers}
\end{aligned}$$

(b)

$$\begin{aligned}
(r^2s^6)^{1/3} &= (r^2)^{1/3}(s^6)^{1/3} && \text{law 3 of exponents}\\
&= r^{2/3}s^2 && \text{law 2 of exponents}
\end{aligned}$$

(c)

$$\begin{aligned}
\left(\frac{2x^{2/3}}{y^{1/2}}\right)^2\left(\frac{3x^{-5/6}}{y^{1/3}}\right) &= \left(\frac{4x^{4/3}}{y}\right)\left(\frac{3x^{-5/6}}{y^{1/3}}\right) && \text{laws of exponents}\\
&= \frac{(4 \cdot 3)x^{4/3-5/6}}{y^{1+(1/3)}} && \text{law 1 of exponents}\\
&= \frac{12x^{8/6-5/6}}{y^{4/3}} && \text{common denominator}\\
&= \frac{12x^{1/2}}{y^{4/3}} && \text{simplify}
\end{aligned}$$

Rational exponents are useful for problems involving radicals that do not have the same index, as illustrated in the next example.

EXAMPLE 6 **Combining radicals**

Change to an expression containing one radical of the form $\sqrt[n]{a^m}$:

(a) $\sqrt[3]{a}\sqrt{a}$ **(b)** $\dfrac{\sqrt[4]{a}}{\sqrt[3]{a^2}}$

SOLUTION Introducing rational exponents, we obtain

(a) $\sqrt[3]{a}\sqrt{a} = a^{1/3}a^{1/2} = a^{(1/3)+(1/2)} = a^{5/6} = \sqrt[6]{a^5}$

(b) $\dfrac{\sqrt[4]{a}}{\sqrt[3]{a^2}} = \dfrac{a^{1/4}}{a^{2/3}} = a^{(1/4)-(2/3)} = a^{-5/12} = \dfrac{1}{a^{5/12}} = \dfrac{1}{\sqrt[12]{a^5}}$

In Exercises 1.2, whenever an index of a radical is even (or a rational exponent m/n with n even is employed), assume that the letters that appear in the radicand denote positive real numbers unless otherwise specified.

1.2 Exercises

Exer. 1–10: Express the number in the form a/b, where a and b are integers.

1 $\left(-\frac{2}{3}\right)^4$

2 $(-3)^3$

3 $\dfrac{2^{-3}}{3^{-2}}$

4 $\dfrac{2^0 + 0^2}{2 + 0}$

5 $-2^4 + 3^{-1}$

6 $\left(-\frac{3}{2}\right)^4 - 2^{-4}$

7 $16^{-3/4}$

8 $9^{5/2}$

9 $(-0.008)^{2/3}$

10 $(0.008)^{-2/3}$

Exer. 11–46: Simplify.

11 $\left(\frac{1}{2}x^4\right)(16x^5)$

12 $(-3x^{-2})(4x^4)$

13 $\dfrac{(2x^3)(3x^2)}{(x^2)^3}$

14 $\dfrac{(2x^2)^3}{4x^4}$

15 $\left(\frac{1}{6}a^5\right)(-3a^2)(4a^7)$

16 $(-4b^3)\left(\frac{1}{6}b^2\right)(-9b^4)$

17 $\dfrac{(6x^3)^2}{(2x^2)^3}\cdot(3x^2)^0$

18 $\dfrac{(3y^3)(2y^2)^2}{(y^4)^3}\cdot(y^3)^0$

19 $(3u^7v^3)(4u^4v^{-5})$

20 $(x^2yz^3)(-2xz^2)(x^3y^{-2})$

21 $(8x^4y^{-3})\left(\frac{1}{2}x^{-5}y^2\right)$

22 $\left(\dfrac{4a^2b}{a^3b^2}\right)\left(\dfrac{5a^2b}{2b^4}\right)$

23 $\left(\frac{1}{3}x^4y^{-3}\right)^{-2}$

24 $(-2xy^2)^5\left(\dfrac{x^7}{8y^3}\right)$

25 $(3y^3)^4(4y^2)^{-3}$

26 $(-3a^2b^{-5})^3$

27 $(-2r^4s^{-3})^{-2}$

28 $(2x^2y^{-5})(6x^{-3}y)\left(\frac{1}{3}x^{-1}y^3\right)$

29 $(5x^2y^{-3})(4x^{-5}y^4)$

30 $(-2r^2s)^5(3r^{-1}s^3)^2$

31 $\left(\dfrac{3x^5y^4}{x^0y^{-3}}\right)^2$

32 $(4a^2b)^4\left(\dfrac{-a^3}{2b}\right)^2$

33 $(4a^{3/2})(2a^{1/2})$

34 $(-6x^{7/5})(2x^{8/5})$

35 $(3x^{5/6})(8x^{2/3})$

36 $(8r)^{1/3}(2r^{1/2})$

37 $(27a^6)^{-2/3}$

38 $(25z^4)^{-3/2}$

39 $(8x^{-2/3})x^{1/6}$

40 $(3x^{1/2})(-2x^{5/2})$

41 $\left(\dfrac{-8x^3}{y^{-6}}\right)^{2/3}$

42 $\left(\dfrac{-y^{3/2}}{y^{-1/3}}\right)^3$

43 $\left(\dfrac{x^6}{9y^{-4}}\right)^{-1/2}$

44 $\left(\dfrac{c^{-4}}{16d^8}\right)^{3/4}$

45 $\dfrac{(x^6y^3)^{-1/3}}{(x^4y^2)^{-1/2}}$

46 $a^{4/3}a^{-3/2}a^{1/6}$

Exer. 47–52: Rewrite the expression using rational exponents.

47 $\sqrt[4]{x^3}$

48 $\sqrt[3]{x^5}$

49 $\sqrt[3]{(a+b)^2}$

50 $\sqrt{a+\sqrt{b}}$

51 $\sqrt{x^2+y^2}$

52 $\sqrt[3]{r^3-s^3}$

Exer. 53–56: Rewrite the expression using a radical.

53 (a) $4x^{3/2}$ (b) $(4x)^{3/2}$

54 (a) $4 + x^{3/2}$ (b) $(4 + x)^{3/2}$

55 (a) $8 - y^{1/3}$ (b) $(8 - y)^{1/3}$

56 (a) $8y^{1/3}$ (b) $(8y)^{1/3}$

Exer. 57–80: Simplify the expression, and rationalize the denominator when appropriate.

57 $\sqrt{81}$

58 $\sqrt[3]{-125}$

59 $\sqrt[5]{-64}$

60 $\sqrt[4]{256}$

61 $\dfrac{1}{\sqrt[3]{2}}$

62 $\sqrt{\dfrac{1}{7}}$

63 $\sqrt{9x^{-4}y^6}$

64 $\sqrt{16a^8b^{-2}}$

65 $\sqrt[3]{8a^6b^{-3}}$

66 $\sqrt[4]{81r^5s^8}$

67 $\sqrt{\dfrac{3x}{2y^3}}$

68 $\sqrt{\dfrac{1}{3x^3y}}$

69 $\sqrt[3]{\dfrac{2x^4y^4}{9x}}$

70 $\sqrt[3]{\dfrac{3x^2y^5}{4x}}$

71 $\sqrt[4]{\dfrac{5x^8y^3}{27x^2}}$

72 $\sqrt[4]{\dfrac{x^7y^{12}}{125x}}$

73 $\sqrt[5]{\dfrac{5x^7y^2}{8x^3}}$

74 $\sqrt[5]{\dfrac{3x^{11}y^3}{9x^2}}$

75 $\sqrt[4]{(3x^5y^{-2})^4}$

76 $\sqrt[6]{(2u^{-3}v^4)^6}$

77 $\sqrt[5]{\dfrac{8x^3}{y^4}}\ \sqrt[5]{\dfrac{4x^4}{y^2}}$

78 $\sqrt{5xy^7}\ \sqrt{10x^3y^3}$

79 $\sqrt[3]{3t^4v^2}\ \sqrt[3]{-9t^{-1}v^4}$

80 $\sqrt[3]{(2r - s)^3}$

Exer. 81–84: Simplify the expression, assuming *x* and *y* may be negative.

81 $\sqrt{x^6y^4}$

82 $\sqrt{x^4y^{10}}$

83 $\sqrt[4]{x^8(y - 1)^{12}}$

84 $\sqrt[4]{(x + 2)^{12}y^4}$

Exer. 85–90: Replace the symbol □ with either = or ≠ to make the resulting statement true, whenever the expression has meaning. Give a reason for your answer.

85 $(a^r)^2 \ \square\ a^{(r^2)}$

86 $(a^2 + 1)^{1/2} \ \square\ a + 1$

87 $a^xb^y \ \square\ (ab)^{xy}$

88 $\sqrt{a^r} \ \square\ \left(\sqrt{a}\right)^r$

89 $\sqrt[n]{\dfrac{1}{c}} \ \square\ \dfrac{1}{\sqrt[n]{c}}$

90 $a^{1/k} \ \square\ \dfrac{1}{a^k}$

Exer. 91–92: In evaluating negative numbers raised to fractional powers, it may be necessary to evaluate the root and integer power separately. For example, $(-3)^{2/5}$ can be evaluated successfully as $[(-3)^{1/5}]^2$ or $[(-3)^2]^{1/5}$, whereas an error message might otherwise appear. Approximate the real-number expression to four decimal places.

91 (a) $(-3)^{2/5}$ (b) $(-5)^{4/3}$

92 (a) $(-1.2)^{3/7}$ (b) $(-5.08)^{7/3}$

Exer. 93–94: Approximate the real-number expression to four decimal places.

93 (a) $\sqrt{\pi + 1}$ (b) $\sqrt[3]{15.1} + 5^{1/4}$

94 (a) $(2.6 - 1.9)^{-2}$ (b) $5^{\sqrt{7}}$

95 **Savings account** One of the oldest banks in the United States is the Bank of America, founded in 1812. If \$200 had been deposited at that time into an account that paid 4% annual interest, then 180 years later the amount would have grown to $200(1.04)^{180}$ dollars. Approximate this amount to the nearest cent.

96 **Viewing distance** On a clear day, the distance d (in miles) that can be seen from the top of a tall building of height h (in feet) can be approximated by $d = 1.2\sqrt{h}$. Approximate the distance that can be seen from the top of the Chicago Sears Tower, which is 1454 feet tall.

97 **Length of a halibut** The length-weight relationship for Pacific halibut can be approximated by the formula $L = 0.46\sqrt[3]{W}$, where W is in kilograms and L is in meters. The largest documented halibut weighed 230 kilograms. Estimate its length.

98 **Weight of a whale** The length-weight relationship for the sei whale can be approximated by $W = 0.0016L^{2.43}$, where W is in tons and L is in feet. Estimate the weight of a whale that is 25 feet long.

99 **Weight lifters' handicaps** O'Carroll's formula is used to handicap weight lifters. If a lifter who weighs b kilograms lifts w kilograms of weight, then the handicapped weight W is given by

$$W = \frac{w}{\sqrt[3]{b - 35}}.$$

Suppose two lifters weighing 75 kilograms and 120 kilograms lift weights of 180 kilograms and 250 kilograms, respectively. Use O'Carroll's formula to determine the superior weight lifter.

100 **Body surface area** A person's body surface area S (in square feet) can be approximated by

$$S = (0.1091)w^{0.425}h^{0.725},$$

where height h is in inches and weight w is in pounds.

(a) Estimate S for a person 6 feet tall weighing 175 pounds.

(b) If a person is 5 feet 6 inches tall, what effect does a 10% increase in weight have on S?

101 **Men's weight** The average weight W (in pounds) for men with height h between 64 and 79 inches can be approximated using the formula $W = 0.1166h^{1.7}$. Construct a table for W by letting $h = 64, 65, \ldots, 79$. Round all weights to the nearest pound.

Height	Weight	Height	Weight
64		72	
65		73	
66		74	
67		75	
68		76	
69		77	
70		78	
71		79	

102 **Women's weight** The average weight W (in pounds) for women with height h between 60 and 75 inches can be approximated using the formula $W = 0.1049h^{1.7}$. Construct a table for W by letting $h = 60, 61, \ldots, 75$. Round all weights to the nearest pound.

Height	Weight	Height	Weight
60		68	
61		69	
62		70	
63		71	
64		72	
65		73	
66		74	
67		75	

1.3

Algebraic Expressions

We sometimes use the notation and terminology of sets to describe mathematical relationships. A **set** is a collection of objects of some type, and the objects are called **elements** of the set. Capital letters $R, S, T, \ldots$ are often used to denote sets, and lowercase letters $a, b, x, y, \ldots$ usually represent elements of sets. Throughout this book, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Z}$ denotes the set of integers.

Two sets S and T are **equal,** denoted by $S = T$, if S and T contain exactly the same elements. We write $S \neq T$ if S and T are not equal. Additional notation and terminology are listed in the following chart.

Notation or terminology	Meaning	Illustrations
$a \in S$	a is an element of S	$3 \in \mathbb{Z}$
$a \notin S$	a is not an element of S	$\frac{3}{5} \notin \mathbb{Z}$
S is a **subset** of T	Every element of S is an element of T	$\mathbb{Z}$ is a subset of $\mathbb{R}$
Constant	A letter or symbol that represents a *specific* element of a set	$5, -\sqrt{2}, \pi$
Variable	A letter or symbol that represents *any* element of a set	Let x denote any real number

We usually use letters near the end of the alphabet, such as x, y, and z, for variables and letters near the beginning of the alphabet, such as a, b, and c, for constants. Throughout this text, unless otherwise specified, variables represent real numbers.

If the elements of a set S have a certain property, we sometimes write $S = \{x: \}$ and state the property describing the variable x in the space after the colon. The expression involving the braces and colon is read "the set of all x such that . . . ," where we complete the phrase by stating the desired property. For example, $\{x: x > 3\}$ is read "the set of all x such that x is greater than 3."

$\{x \mid x > 3\}$ is an equivalent notation.

For finite sets, we sometimes list all the elements of the set within braces. Thus, if the set T consists of the first five positive integers, we may write $T = \{1, 2, 3, 4, 5\}$. When we describe sets in this way, the order used in listing the elements is irrelevant, so we could also write $T = \{1, 3, 2, 4, 5\}$, $T = \{4, 3, 2, 5, 1\}$, and so on.

If we begin with any collection of variables and real numbers, then an **algebraic expression** is the result obtained by applying additions, subtractions, multiplications, divisions, powers, or the taking of roots to this collection. If specific numbers are substituted for the variables in an algebraic expression, the resulting number is called the **value** of the expression for these numbers. The **domain** of an algebraic expression consists of all real numbers that may represent the variables. Thus, unless otherwise specified, *we assume that the domain consists of the real numbers that, when substituted for the variables, do not make the expression meaningless, in the sense that denominators cannot equal zero and roots always exist.* Two illustrations are given in the following chart.

Algebraic Expressions

Illustration	Domain	Typical value
$x^3 - 5x + \dfrac{6}{\sqrt{x}}$	all $x > 0$	At $x = 4$: $4^3 - 5(4) + \dfrac{6}{\sqrt{4}} = 64 - 20 + 3 = 47$
$\dfrac{2xy + (3/x^2)}{\sqrt[3]{y - 1}}$	all $x \neq 0$ and all $y \neq 1$	At $x = 1$ and $y = 9$: $\dfrac{2(1)(9) + (3/1^2)}{\sqrt[3]{9 - 1}} = \dfrac{18 + 3}{\sqrt[3]{8}} = \dfrac{21}{2}$

If x is a variable, then a **monomial** in x is an expression of the form ax^n, where a is a real number and n is a nonnegative integer. A **binomial** is a sum of two monomials, and a **trinomial** is a sum of three monomials. A *polynomial in x* is a sum of any number of monomials in x. Another way of stating this is as follows.

Definition of Polynomial

A **polynomial in *x*** is a sum of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where n is a nonnegative integer and each coefficient a_k is a real number. If $a_n \neq 0$, then the polynomial is said to have **degree *n*.**

Each expression a_kx^k in the sum is a **term** of the polynomial. If a coefficient a_k is zero, we usually delete the term a_kx^k. The coefficient a_k of the highest power of x is called the **leading coefficient** of the polynomial.

The following chart contains specific illustrations of polynomials.

Polynomials

Example	Leading coefficient	Degree
$3x^4 + 5x^3 + (-7)x + 4$	3	4
$x^8 + 9x^2 + (-2)x$	1	8
$-5x^2 + 1$	-5	2
$7x + 2$	7	1
8	8	0

By definition, two polynomials are **equal** if and only if they have the same degree and the coefficients of like powers of x are equal. If all the coefficients of a polynomial are zero, it is called the **zero polynomial** and is denoted by 0. However, by convention, the degree of the zero polynomial is *not* zero but, instead, is undefined. If c is a *nonzero real number,* then c is a polynomial of degree 0. Such polynomials (together with the zero polynomial) are **constant polynomials.**

If a coefficient of a polynomial is negative, we usually use a minus sign between appropriate terms. To illustrate,

$$3x^2 + (-5)x + (-7) = 3x^2 - 5x - 7.$$

We may also consider polynomials in variables other than x. For example, $\frac{2}{5}z^2 - 3z^7 + 8 - \sqrt{5}z^4$ is a polynomial in z of degree 7. We often arrange the terms of a polynomial in order of decreasing powers of the variable; thus, we write

$$\tfrac{2}{5}z^2 - 3z^7 + 8 - \sqrt{5}z^4 = -3z^7 - \sqrt{5}z^4 + \tfrac{2}{5}z^2 + 8.$$

We may regard a polynomial in x as an algebraic expression obtained by employing a finite number of additions, subtractions, and multiplications involving x. If an algebraic expression contains divisions or roots involving a variable x, then it is not a polynomial in x.

ILLUSTRATION **Nonpolynomials**

- $\dfrac{1}{x} + 3x$
- $\dfrac{x - 5}{x^2 + 2}$
- $3x^2 + \sqrt{x} - 2$

Since polynomials represent real numbers, we may use the properties described in Section 1.1. In particular, if additions, subtractions, and multiplications are carried out with polynomials, we may simplify the results by using properties of real numbers, as demonstrated in the following examples.

EXAMPLE 1 **Adding and subtracting polynomials**

(a) Find the sum: $(x^3 + 2x^2 - 5x + 7) + (4x^3 - 5x^2 + 3)$

(b) Find the difference: $(x^3 + 2x^2 - 5x + 7) - (4x^3 - 5x^2 + 3)$

SOLUTION

(a) To obtain the sum of any two polynomials in x, we may add coefficients of like powers of x.

$$\begin{aligned}
&(x^3 + 2x^2 - 5x + 7) + (4x^3 - 5x^2 + 3) \\
&\quad = x^3 + 2x^2 - 5x + 7 + 4x^3 - 5x^2 + 3 && \text{remove parentheses} \\
&\quad = (1 + 4)x^3 + (2 - 5)x^2 - 5x + (7 + 3) && \text{add coefficients of like powers of } x \\
&\quad = 5x^3 - 3x^2 - 5x + 10 && \text{simplify}
\end{aligned}$$

The grouping in the first step was shown for completeness. You may omit this step after you become proficient with such manipulations.

(b) When subtracting polynomials, we first remove parentheses, noting that the minus sign preceding the second pair of parentheses changes the sign of *each* term of that polynomial.

$$\begin{aligned}
&(x^3 + 2x^2 - 5x + 7) - (4x^3 - 5x^2 + 3) \\
&\quad = x^3 + 2x^2 - 5x + 7 - 4x^3 + 5x^2 - 3 && \text{remove parentheses} \\
&\quad = (1 - 4)x^3 + (2 + 5)x^2 - 5x + (7 - 3) && \text{add coefficients of like powers of } x \\
&\quad = -3x^3 + 7x^2 - 5x + 4 && \text{simplify}
\end{aligned}$$

EXAMPLE 2 **Multiplying binomials**

Find the product: $(4x + 5)(3x - 2)$

SOLUTION Since $3x - 2 = 3x + (-2)$, we may proceed as in Example 1 of Section 1.1:

Calculator check for Example 2: Store 17 in a memory location and show that the original expression and the final expression both equal 3577.

$$\begin{aligned}
&(4x + 5)(3x - 2) \\
&\quad = (4x)(3x) + (4x)(-2) + (5)(3x) + (5)(-2) && \text{distributive properties} \\
&\quad = 12x^2 - 8x + 15x - 10 && \text{multiply} \\
&\quad = 12x^2 + 7x - 10 && \text{simplify}
\end{aligned}$$

After becoming proficient working problems of the type in Example 2, you may wish to perform the first two steps mentally and proceed directly to the final form.

In the next example we illustrate different methods for finding the product of two polynomials.

EXAMPLE 3 **Multiplying polynomials**

Find the product: $(x^2 + 5x - 4)(2x^3 + 3x - 1)$

SOLUTION

Method 1 We begin by using a distributive property, treating the polynomial $2x^3 + 3x - 1$ as a single real number:

$$\begin{aligned}&(x^2 + 5x - 4)(2x^3 + 3x - 1)\\ &\quad = x^2(2x^3 + 3x - 1) + 5x(2x^3 + 3x - 1) - 4(2x^3 + 3x - 1)\end{aligned}$$

We next use another distributive property three times and simplify the result, obtaining

$$\begin{aligned}&(x^2 + 5x - 4)(2x^3 + 3x - 1)\\ &\quad = 2x^5 + 3x^3 - x^2 + 10x^4 + 15x^2 - 5x - 8x^3 - 12x + 4\\ &\quad = 2x^5 + 10x^4 - 5x^3 + 14x^2 - 17x + 4.\end{aligned}$$

Note that the three monomials in the first polynomial were multiplied by each of the three monomials in the second polynomial, giving us a total of nine terms.

Method 2 We list the polynomials vertically and multiply, leaving spaces for powers of x that have zero coefficients, as follows:

$$\begin{array}{llllll}
2x^3 & +\ 3x & -\ 1 & & & \\
x^2 & +\ 5x & -\ 4 & & & \\
\hline
2x^5 & & +\ 3x^3 & -\ x^2 & & & = x^2(2x^3 + 3x - 1)\\
 & 10x^4 & & +\ 15x^2 & -\ 5x & & = 5x(2x^3 + 3x - 1)\\
 & & -\ 8x^3 & & -\ 12x & +\ 4 & = -4(2x^3 + 3x - 1)\\
\hline
2x^5 & +\ 10x^4 & -\ 5x^3 & +\ 14x^2 & -\ 17x & +\ 4 & = \text{sum of the above}
\end{array}$$

In practice, we would omit the reasons (equalities) listed on the right in the last four lines.

We may consider polynomials in more than one variable. For example, a polynomial in *two* variables, x and y, is a finite sum of terms, each of the form ax^my^k for some real number a and nonnegative integers m and k. An example is

$$3x^4y + 2x^3y^5 + 7x^2 - 4xy + 8y - 5.$$

Other polynomials may involve three variables—such as x, y, z—or, for that matter, *any* number of variables. Addition, subtraction, and multiplication are performed using properties of real numbers, just as for polynomials in one variable.

The next example illustrates division of a polynomial by a monomial.

EXAMPLE 4 **Dividing a polynomial by a monomial**

Express as a polynomial in x and y:

$$\frac{6x^2y^3 + 4x^3y^2 - 10xy}{2xy}$$

SOLUTION

$$\frac{6x^2y^3 + 4x^3y^2 - 10xy}{2xy} = \frac{6x^2y^3}{2xy} + \frac{4x^3y^2}{2xy} - \frac{10xy}{2xy} \qquad \text{divide each term by } 2xy$$

$$= 3xy^2 + 2x^2y - 5 \qquad \text{simplify}$$

The products listed in the next chart occur so frequently that they deserve special attention. You can check the validity of each formula by multiplication. In (2) and (3), we use either the top sign on both sides or the bottom sign on both sides. Thus, (2) is actually *two* formulas:

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad \text{and} \qquad (x - y)^2 = x^2 - 2xy + y^2$$

Similarly, (3) represents two formulas.

Product Formulas

Formula	Illustration
(1) $(x + y)(x - y) = x^2 - y^2$	$(2a + 3)(2a - 3) = (2a)^2 - 3^2 = 4a^2 - 9$
(2) $(x \pm y)^2 = x^2 \pm 2xy + y^2$	$(2a - 3)^2 = (2a)^2 - 2(2a)(3) + (3)^2$ $= 4a^2 - 12a + 9$
(3) $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$	$(2a + 3)^3 = (2a)^3 + 3(2a)^2(3) + 3(2a)(3)^2 + (3)^3$ $= 8a^3 + 36a^2 + 54a + 27$

Several other illustrations of the product formulas are given in the next example.

EXAMPLE 5 **Using product formulas**

Find the product:

(a) $(2r^2 - \sqrt{s})(2r^2 + \sqrt{s})$ **(b)** $\left(\sqrt{c} + \dfrac{1}{\sqrt{c}}\right)^2$ **(c)** $(2a - 5b)^3$

SOLUTION

(a) We use product formula 1, with $x = 2r^2$ and $y = \sqrt{s}$:

$$\begin{aligned}(2r^2 - \sqrt{s})(2r^2 + \sqrt{s}) &= (2r^2)^2 - (\sqrt{s})^2 \\ &= 4r^4 - s\end{aligned}$$

(b) We use product formula 2, with $x = \sqrt{c}$ and $y = \dfrac{1}{\sqrt{c}}$:

$$\begin{aligned}\left(\sqrt{c} + \frac{1}{\sqrt{c}}\right)^2 &= (\sqrt{c})^2 + 2 \cdot \sqrt{c} \cdot \frac{1}{\sqrt{c}} + \left(\frac{1}{\sqrt{c}}\right)^2 \\ &= c + 2 + \frac{1}{c}\end{aligned}$$

Note that the last expression is *not* a polynomial.

(c) We use product formula 3, with $x = 2a$ and $y = 5b$:

$$\begin{aligned}(2a - 5b)^3 &= (2a)^3 - 3(2a)^2(5b) + 3(2a)(5b)^2 - (5b)^3 \\ &= 8a^3 - 60a^2b + 150ab^2 - 125b^3\end{aligned}$$

If a polynomial is a product of other polynomials, then each polynomial in the product is a **factor** of the original polynomial. **Factoring** is the process of expressing a sum of terms as a product. For example, since $x^2 - 9 = (x + 3)(x - 3)$, the polynomials $x + 3$ and $x - 3$ are factors of $x^2 - 9$.

Factoring is an important process in mathematics, since it may be used to reduce the study of a complicated expression to the study of several simpler expressions. For example, properties of the polynomial $x^2 - 9$ can be determined by examining the factors $x + 3$ and $x - 3$. As we shall see in Chapter 2, another important use for factoring is in finding solutions of equations.

We shall be interested primarily in **nontrivial factors** of polynomials—that is, factors that contain polynomials of positive degree. However, if the coefficients are restricted to *integers,* then we usually remove a common integral factor from each term of the polynomial. For example,

$$4x^2y + 8z^3 = 4(x^2y + 2z^3).$$

A polynomial with coefficients in some set S of numbers is **prime,** or **irreducible** over S, if it cannot be written as a product of two polynomials of positive degree with coefficients in S. A polynomial may be irreducible over one set S but not over another. For example, $x^2 - 2$ is irreducible over the rational numbers, since it cannot be expressed as a product of two polynomials of positive degree that have *rational* coefficients. However, $x^2 - 2$ is *not* irreducible over the real numbers, since we can write

$$x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2}).$$

Similarly, $x^2 + 1$ is irreducible over the real numbers, but, as we shall see in Section 2.4, not over the complex numbers.

Every polynomial $ax + b$ of degree 1 is irreducible.

Before we factor a polynomial, we must specify the number system (or set) from which the coefficients of the factors are to be chosen. In this chapter we shall use the rule that *if a polynomial has integral coefficients, then the factors should be polynomials with integral coefficients*. To **factor a polynomial** means to express it as a product of irreducible polynomials.

The **greatest common factor (gcf)** of an expression is the product of the factors that appear in each term, with each of these factors raised to the smallest nonzero exponent appearing in any term. In factoring polynomials, it is advisable to first factor out the gcf, as shown in the following illustration.

ILLUSTRATION **Factored Polynomials**

- $8x^2 + 4xy = 4x(2x + y)$
- $25x^2 + 25x - 150 = 25(x^2 + x - 6) = 25(x + 3)(x - 2)$
- $4x^5y - 9x^3y^3 = x^3y(4x^2 - 9y^2) = x^3y(2x + 3y)(2x - 3y)$

It is usually difficult to factor polynomials of degree greater than 2. In simple cases, the following factoring formulas may be useful. Each formula can be verified by multiplying the factors on the right-hand side of the equals sign. It can be shown that the factors $x^2 + xy + y^2$ and $x^2 - xy + y^2$ in the difference and sum of two cubes, respectively, are irreducible over the real numbers.

Factoring Formulas

Formula	Illustration
(1) Difference of two squares: $x^2 - y^2 = (x + y)(x - y)$	$9a^2 - 16 = (3a)^2 - (4)^2 = (3a + 4)(3a - 4)$
(2) Difference of two cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$	$8a^3 - 27 = (2a)^3 - (3)^3 = (2a - 3)[(2a)^2 + (2a)(3) + (3)^2] = (2a - 3)(4a^2 + 6a + 9)$
(3) Sum of two cubes: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$	$125a^3 + 1 = (5a)^3 + (1)^3 = (5a + 1)[(5a)^2 - (5a)(1) + (1)^2] = (5a + 1)(25a^2 - 5a + 1)$

Several other illustrations of the use of factoring formulas are given in the next two examples.

EXAMPLE 6 **Difference of two squares**

Factor each polynomial:

(a) $25r^2 - 49s^2$ **(b)** $81x^4 - y^4$ **(c)** $16x^4 - (y - 2z)^2$

SOLUTION

(a) We apply the difference of two squares formula, with $x = 5r$ and $y = 7s$:

$$25r^2 - 49s^2 = (5r)^2 - (7s)^2 = (5r + 7s)(5r - 7s)$$

(b) We write $81x^4 = (9x^2)^2$ and $y^4 = (y^2)^2$ and apply the difference of two squares formula twice:

$$\begin{aligned} 81x^4 - y^4 &= (9x^2)^2 - (y^2)^2 \\ &= (9x^2 + y^2)(9x^2 - y^2) \\ &= (9x^2 + y^2)[(3x)^2 - (y)^2] \\ &= (9x^2 + y^2)(3x + y)(3x - y) \end{aligned}$$

(c) We write $16x^4 = (4x^2)^2$ and apply the difference of two squares formula:

$$\begin{aligned} 16x^4 - (y - 2z)^2 &= (4x^2)^2 - (y - 2z)^2 \\ &= [(4x^2) + (y - 2z)][(4x^2) - (y - 2z)] \\ &= (4x^2 + y - 2z)(4x^2 - y + 2z) \end{aligned}$$

EXAMPLE 7 **Sum and difference of two cubes**

Factor each polynomial:

(a) $a^3 + 64b^3$ **(b)** $8c^6 - 27d^9$

SOLUTION

(a) We apply the sum of two cubes formula, with $x = a$ and $y = 4b$:

$$\begin{aligned} a^3 + 64b^3 &= a^3 + (4b)^3 \\ &= (a + 4b)[a^2 - a(4b) + (4b)^2] \\ &= (a + 4b)(a^2 - 4ab + 16b^2) \end{aligned}$$

(b) We apply the difference of two cubes formula, with $x = 2c^2$ and $y = 3d^3$:

$$\begin{aligned} 8c^6 - 27d^9 &= (2c^2)^3 - (3d^3)^3 \\ &= (2c^2 - 3d^3)[(2c^2)^2 + (2c^2)(3d^3) + (3d^3)^2] \\ &= (2c^2 - 3d^3)(4c^4 + 6c^2d^3 + 9d^6) \end{aligned}$$

A factorization of a trinomial $px^2 + qx + r$, where p, q, and r are integers, must be of the form

$$px^2 + qx + r = (ax + b)(cx + d),$$

where a, b, c, and d are integers. It follows that

$$ac = p, \qquad bd = r, \qquad \text{and} \qquad ad + bc = q.$$

Only a limited number of choices for a, b, c, and d satisfy these conditions. If none of the choices work, then $px^2 + qx + r$ is irreducible. Trying the various possibilities, as depicted in the next example, is called the **method of trial and error.** This method is also applicable to trinomials of the form $px^2 + qxy + ry^2$, in which case the factorization must be of the form $(ax + by)(cx + dy)$.

EXAMPLE 8 **Factoring a trinomial by trial and error**

Factor $6x^2 - 7x - 3$.

SOLUTION If we write

$$6x^2 - 7x - 3 = (ax + b)(cx + d),$$

then the following relationships must be true:

$$ac = 6, \qquad bd = -3, \qquad \text{and} \qquad ad + bc = -7$$

If we assume that a and c are both positive, then all possible values are given in the following table:

a	1	6	2	3
c	6	1	3	2

Thus, if $6x^2 - 7x - 3$ is factorable, then one of the following is true:

$$6x^2 - 7x - 3 = (x + b)(6x + d)$$
$$6x^2 - 7x - 3 = (6x + b)(x + d)$$
$$6x^2 - 7x - 3 = (2x + b)(3x + d)$$
$$6x^2 - 7x - 3 = (3x + b)(2x + d)$$

We next consider all possible values for b and d. Since $bd = -3$, these are as follows:

b	1	−1	3	−3
d	−3	3	−1	1

Trying various (possibly all) values, we arrive at $b = -3$ and $d = 1$; that is,

$$6x^2 - 7x - 3 = (2x - 3)(3x + 1).$$

As a check, you should multiply the final factorization to see whether the original polynomial is obtained.

The method of trial and error illustrated in Example 8 can be long and tedious if the coefficients of the polynomial are large and have many prime factors. We will show a factoring method in Section 2.3 that can be used to factor any trinomial of the form of the one in Example 8—regardless of the size of the coefficients. For simple cases, it is often possible to arrive at the correct choice rapidly.

EXAMPLE 9 Factoring polynomials

Factor:

(a) $12x^2 - 36xy + 27y^2$ **(b)** $4x^4y - 11x^3y^2 + 6x^2y^3$

SOLUTION

(a) Since each term has 3 as a factor, we begin by writing

$$12x^2 - 36xy + 27y^2 = 3(4x^2 - 12xy + 9y^2).$$

A factorization of $4x^2 - 12xy + 9y^2$ as a product of two first-degree polynomials must be of the form

$$4x^2 - 12xy + 9y^2 = (ax + by)(cx + dy),$$

with $ac = 4,\quad bd = 9,\quad$ and $\quad ad + bc = -12.$

Using the method of trial and error, as in Example 8, we obtain

$$4x^2 - 12xy + 9y^2 = (2x - 3y)(2x - 3y) = (2x - 3y)^2.$$

Thus, $12x^2 - 36xy + 27y^2 = 3(4x^2 - 12xy + 9y^2) = 3(2x - 3y)^2$.

(b) Since each term has x^2y as a factor, we begin by writing

$$4x^4y - 11x^3y^2 + 6x^2y^3 = x^2y(4x^2 - 11xy + 6y^2).$$

By trial and error, we obtain the factorization

$$4x^4y - 11x^3y^2 + 6x^2y^3 = x^2y(4x - 3y)(x - 2y).$$

If a sum contains four or more terms, it may be possible to group the terms in a suitable manner and then find a factorization by using distributive properties. This technique, called **factoring by grouping,** is illustrated in the next example.

EXAMPLE 10 **Factoring by grouping**

Factor:

(a) $4ac + 2bc - 2ad - bd$ **(b)** $3x^3 + 2x^2 - 12x - 8$

(c) $x^2 - 16y^2 + 10x + 25$

SOLUTION

(a) We group the first two terms and the last two terms and then proceed as follows:

$$\begin{aligned} 4ac + 2bc - 2ad - bd &= (4ac + 2bc) - (2ad + bd) \\ &= 2c(2a + b) - d(2a + b) \end{aligned}$$

At this stage we have not factored the given expression because the right-hand side has the form

$$2ck - dk \quad \text{with } k = 2a + b.$$

However, if we factor out k, then

$$2ck - dk = (2c - d)k = (2c - d)(2a + b).$$

Hence,

$$\begin{aligned} 4ac + 2bc - 2ad - bd &= 2c(2a + b) - d(2a + b) \\ &= (2c - d)(2a + b). \end{aligned}$$

Note that if we factor $2ck - dk$ as $k(2c - d)$, then the last expression is $(2a + b)(2c - d)$.

(b) We group the first two terms and the last two terms and then proceed as follows:

$$\begin{aligned} 3x^3 + 2x^2 - 12x - 8 &= (3x^3 + 2x^2) - (12x + 8) \\ &= x^2(3x + 2) - 4(3x + 2) \\ &= (x^2 - 4)(3x + 2) \end{aligned}$$

Finally, using the difference of two squares formula for $x^2 - 4$, we obtain the factorization:

$$3x^3 + 2x^2 - 12x - 8 = (x + 2)(x - 2)(3x + 2)$$

(c) First we rearrange and group terms, and then we apply the difference of two squares formula, as follows:

$$\begin{aligned} x^2 - 16y^2 + 10x + 25 &= (x^2 + 10x + 25) - 16y^2 \\ &= (x + 5)^2 - (4y)^2 \\ &= [(x + 5) + 4y][(x + 5) - 4y] \\ &= (x + 4y + 5)(x - 4y + 5) \end{aligned}$$

1.3 Exercises

Exer. 1–44: Express as a polynomial.

1 $(3x^3 + 4x^2 - 7x + 1) + (9x^3 - 4x^2 - 6x)$

2 $(7x^3 + 2x^2 - 11x) + (-3x^3 - 2x^2 + 5x - 3)$

3 $(4x^3 + 5x - 3) - (3x^3 + 2x^2 + 5x - 7)$

4 $(6x^3 - 2x^2 + x - 2) - (8x^2 - x - 2)$

5 $(2x + 5)(3x - 7)$

6 $(3x - 4)(2x + 9)$

7 $(5x + 7y)(3x + 2y)$

8 $(4x - 3y)(x - 5y)$

9 $(2u + 3)(u - 4) + 4u(u - 2)$

10 $(3u - 1)(u + 2) + 7u(u + 1)$

11 $(3x + 5)(2x^2 + 9x - 5)$

12 $(7x - 4)(x^3 - x^2 + 6)$

13 $(t^2 + 2t - 5)(3t^2 - t + 2)$

14 $(r^2 - 8r - 2)(-r^2 + 3r - 1)$

15 $(x + 1)(2x^2 - 2)(x^3 + 5)$

16 $(2x - 1)(x^2 - 5)(x^3 - 1)$

17 $\dfrac{8x^2y^3 - 10x^3y}{2x^2y}$

18 $\dfrac{6a^3b^3 - 9a^2b^2 + 3ab^4}{3ab^2}$

19 $\dfrac{3u^3v^4 - 2u^5v^2 + (u^2v^2)^2}{u^3v^2}$

20 $\dfrac{6x^2yz^3 - xy^2z}{xyz}$

21 $(2x + 3y)(2x - 3y)$

22 $(5x + 4y)(5x - 4y)$

23 $(x^2 + 2y)(x^2 - 2y)$

24 $(3x + y^3)(3x - y^3)$

25 $(x^2 + 9)(x^2 - 4)$

26 $(x^2 + 1)(x^2 - 16)$

27 $(3x + 2y)^2$

28 $(5x - 4y)^2$

29 $(x^2 - 3y^2)^2$

30 $(2x^2 + 5y^2)^2$

31 $(x + 2)^2(x - 2)^2$

32 $(x + y)^2(x - y)^2$

33 $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$

34 $(\sqrt{x} + \sqrt{y})^2(\sqrt{x} - \sqrt{y})^2$

35 $(x^{1/3} - y^{1/3})(x^{2/3} + x^{1/3}y^{1/3} + y^{2/3})$

36 $(x^{1/3} + y^{1/3})(x^{2/3} - x^{1/3}y^{1/3} + y^{2/3})$

37 $(x - 2y)^3$

38 $(x + 3y)^3$

39 $(2x + 3y)^3$

40 $(3x - 4y)^3$

41 $(a + b - c)^2$

42 $(x^2 + x + 1)^2$

43 $(2x + y - 3z)^2$

44 $(x - 2y + 3z)^2$

Exer. 45–102: Factor the polynomial.

45 $rs + 4st$

46 $4u^2 - 2uv$

47 $3a^2b^2 - 6a^2b$

48 $10xy + 15xy^2$

49 $3x^2y^3 - 9x^3y^2$

50 $16x^5y^2 + 8x^3y^3$

51 $15x^3y^5 - 25x^4y^2 + 10x^6y^4$

52 $121r^3s^4 + 77r^2s^4 - 55r^4s^3$

53 $8x^2 - 53x - 21$

54 $7x^2 + 10x - 8$

55 $x^2 + 3x + 4$

56 $3x^2 - 4x + 2$

57 $6x^2 + 7x - 20$

58 $12x^2 - x - 6$

59 $12x^2 - 29x + 15$

60 $21x^2 + 41x + 10$

61 $4x^2 - 20x + 25$

62 $9x^2 + 24x + 16$

63 $25z^2 + 30z + 9$

64 $16z^2 - 56z + 49$

65 $45x^2 + 38xy + 8y^2$

66 $50x^2 + 45xy - 18y^2$

67 $36r^2 - 25t^2$

68 $81r^2 - 16t^2$

69 $z^4 - 64w^2$

70 $9y^4 - 121x^2$

71 $x^4 - 4x^2$

72 $x^3 - 25x$

73 $x^2 + 25$

74 $4x^2 + 9$

75 $75x^2 - 48y^2$

76 $64x^2 - 36y^2$

77 $64x^3 + 27$

78 $125x^3 - 8$

79 $64x^3 - y^6$

80 $216x^9 + 125y^3$

81 $343x^3 + y^9$

82 $x^6 - 27y^3$

83 $125 - 27x^3$

84 $x^3 + 64$

85 $2ax - 6bx + ay - 3by$

86 $2ay^2 - axy + 6xy - 3x^2$

87 $3x^3 + 3x^2 - 27x - 27$

88 $5x^3 + 10x^2 - 20x - 40$

89 $x^4 + 2x^3 - x - 2$

90 $x^4 - 3x^3 + 8x - 24$

91 $a^3 - a^2b + ab^2 - b^3$

92 $6w^8 + 17w^4 + 12$

93 $a^6 - b^6$

94 $x^8 - 16$

95 $x^2 + 4x + 4 - 9y^2$

96 $x^2 - 4y^2 - 6x + 9$

97 $y^2 - x^2 + 8y + 16$

98 $y^2 + 9 - 6y - 4x^2$

99 $y^6 + 7y^3 - 8$

100 $8c^6 + 19c^3 - 27$

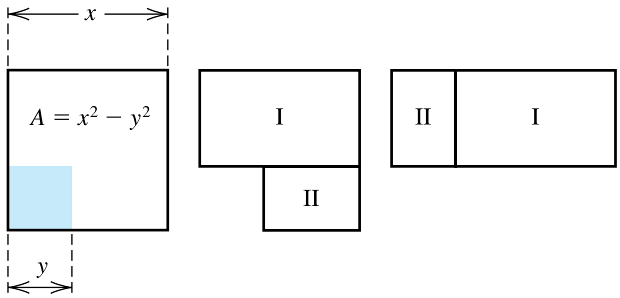
101 $x^{16} - 1$

102 $4x^3 + 4x^2 + x$

Exer. 103–104: The ancient Greeks gave geometric proofs of the factoring formulas for the difference of two squares and the difference of two cubes. Establish the formula for the special case described.

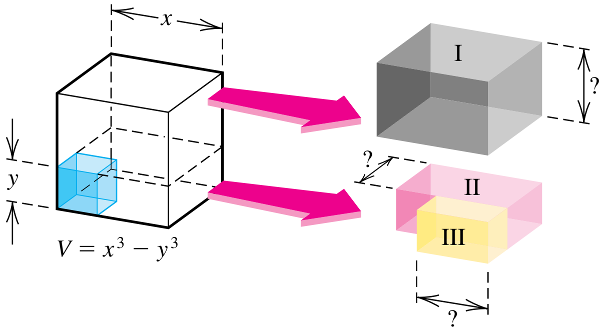
103 Find the areas of regions I and II in the figure to establish the difference of two squares formula for the special case $x > y$.

Exercise 103



104 Find the volumes of boxes I, II, and III in the figure to establish the difference of two cubes formula for the special case $x > y$.

Exercise 104



105 **Calorie requirements** The basal energy requirement for an individual indicates the minimum number of calories necessary to maintain essential life-sustaining processes such as circulation, regulation of body temperature, and respiration. Given a person's sex, weight w (in kilograms), height h (in centimeters), and age y (in years), we can estimate the basal energy requirement in calories using the following formulas, where C_f and C_m are the calories necessary for females and males, respectively:

$$C_f = 66.5 + 13.8w + 5h - 6.8y$$
$$C_m = 655 + 9.6w + 1.9h - 4.7y$$

(a) Determine the basal energy requirements first for a 25-year-old female weighing 59 kilograms who is 163 centimeters tall and then for a 55-year-old male weighing 75 kilograms who is 178 centimeters tall.

(b) Discuss why, in both formulas, the coefficient for y is negative but the other coefficients are positive.

1.4 *Fractional Expressions*

A **fractional expression** is a quotient of two algebraic expressions. As a special case, a **rational expression** is a quotient p/q of two *polynomials* p and q. Since division by zero is not allowed, the domain of p/q consists of all real numbers except those that make the denominator zero. Two illustrations are given in the chart.

Rational Expressions

Quotient	Denominator is zero if	Domain
$\dfrac{6x^2 - 5x + 4}{x^2 - 9}$	$x = \pm 3$	All $x \neq \pm 3$
$\dfrac{x^3 - 3x^2y + 4y^2}{y - x^3}$	$y = x^3$	All x and y such that $y \neq x^3$

In most of our work we will be concerned with rational expressions in which both numerator and denominator are polynomials in only one variable.

Since the variables in a rational expression represent real numbers, we may use the properties of quotients in Section 1.1, replacing the letters a, b, c, and d with polynomials. The following property is of particular importance, where $bd \neq 0$:

$$\frac{ad}{bd} = \frac{a}{b} \cdot \frac{d}{d} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

We sometimes describe this simplification process by saying that *a common nonzero factor in the numerator and denominator of a quotient may be canceled.* In practice, we usually show this cancellation by means of a slash through the common factor, as in the following illustration, where all denominators are assumed to be nonzero.

ILLUSTRATION Canceled Common Factors

■ $\dfrac{a\cancel{d}}{b\cancel{d}} = \dfrac{a}{b}$ ■ $\dfrac{m\cancel{n}}{\cancel{n}pq} = \dfrac{m}{pq}$ ■ $\dfrac{\cancel{p}q\cancel{r}}{\cancel{r}\cancel{p}v} = \dfrac{q}{v}$

A rational expression is *simplified,* or *reduced to lowest terms,* if the numerator and denominator have no common polynomial factors of positive degree and no common integral factors greater than 1. To simplify a rational expression, we factor both the numerator and the denominator into prime factors and then, assuming the factors in the denominator are not zero, cancel common factors, as in the following illustration.

ILLUSTRATION Simplified Rational Expressions

■ $\dfrac{3x^2 - 5x - 2}{x^2 - 4} = \dfrac{(3x + 1)\cancel{(x - 2)}}{(x + 2)\cancel{(x - 2)}} \overset{\text{if } x \neq 2}{=} \dfrac{3x + 1}{x + 2}$

■ $\dfrac{2 - x - 3x^2}{6x^2 - x - 2} = \dfrac{-(3x^2 + x - 2)}{6x^2 - x - 2} = -\dfrac{\cancel{(3x - 2)}(x + 1)}{\cancel{(3x - 2)}(2x + 1)} \overset{\text{if } x \neq 2/3}{=} -\dfrac{x + 1}{2x + 1}$

■ $\dfrac{(x^2 + 8x + 16)(x - 5)}{(x^2 - 5x)(x^2 - 16)} = \dfrac{(x + 4)^{\cancel{2}\,1}\cancel{(x - 5)}}{x\cancel{(x - 5)}\cancel{(x + 4)}(x - 4)} \overset{\text{if } x \neq 5,\, x \neq -4}{=} \dfrac{x + 4}{x(x - 4)}$

As shown in the next example, when simplifying a product or quotient of rational expressions, we often use properties of quotients to obtain one rational expression. Then we factor the numerator and denominator and cancel common factors, as we did in the preceding illustration.

EXAMPLE 1 **Products and quotients of rational expressions**

Perform the indicated operation and simplify:

(a) $\dfrac{x^2 - 6x + 9}{x^2 - 1} \cdot \dfrac{2x - 2}{x - 3}$ (b) $\dfrac{x + 2}{2x - 3} \div \dfrac{x^2 - 4}{2x^2 - 3x}$

SOLUTION

(a)
$$\frac{x^2 - 6x + 9}{x^2 - 1} \cdot \frac{2x - 2}{x - 3} = \frac{(x^2 - 6x + 9)(2x - 2)}{(x^2 - 1)(x - 3)} \quad \text{property of quotients}$$

$$= \frac{(x - 3)^{\cancel{2}\,1} \cdot 2\cancel{(x - 1)}}{(x + 1)\cancel{(x - 1)}\cancel{(x - 3)}} \quad \text{factor all polynomials}$$

if $x \neq 3, x \neq 1$

$$\overset{\downarrow}{=} \frac{2(x - 3)}{x + 1} \quad \text{cancel common factors}$$

(b)
$$\frac{x + 2}{2x - 3} \div \frac{x^2 - 4}{2x^2 - 3x} = \frac{x + 2}{2x - 3} \cdot \frac{2x^2 - 3x}{x^2 - 4} \quad \text{property of quotients}$$

$$= \frac{\cancel{(x + 2)}x\cancel{(2x - 3)}}{\cancel{(2x - 3)}\cancel{(x + 2)}(x - 2)} \quad \text{property of quotients; factor all polynomials}$$

if $x \neq -2, x \neq 3/2$

$$\overset{\downarrow}{=} \frac{x}{x - 2} \quad \text{cancel common factors}$$

To add or subtract two rational expressions, we usually find a *common denominator* and use the following properties of quotients:

$$\frac{a}{d} + \frac{c}{d} = \frac{a + c}{d} \quad \text{and} \quad \frac{a}{d} - \frac{c}{d} = \frac{a - c}{d}$$

If the denominators of the expressions are not the same, we may obtain a common denominator by multiplying the numerator and denominator of each fraction by a suitable expression. We usually use the ***least* common denominator (lcd)** of the two quotients. To find the lcd, we factor each denominator into primes and then form the product of the different prime factors, using the *largest* exponent that appears with each prime factor. Let us begin with a numerical example of this technique.

EXAMPLE 2 **Adding fractions using the lcd**

Express as a simplified rational number:

$$\frac{7}{24} + \frac{5}{18}$$

SOLUTION The prime factorizations of the denominators 24 and 18 are $24 = 2^3 \cdot 3$ and $18 = 2 \cdot 3^2$. To find the lcd, we form the product of the different prime factors, using the largest exponent associated with each factor. This gives us $2^3 \cdot 3^2$. We now change each fraction to an equivalent fraction with denominator $2^3 \cdot 3^2$ and add:

$$\begin{aligned}
\frac{7}{24} + \frac{5}{18} &= \frac{7}{2^3 \cdot 3} + \frac{5}{2 \cdot 3^2} \\
&= \frac{7}{2^3 \cdot 3} \cdot \frac{3}{3} + \frac{5}{2 \cdot 3^2} \cdot \frac{2^2}{2^2} \\
&= \frac{21}{2^3 \cdot 3^2} + \frac{20}{2^3 \cdot 3^2} \\
&= \frac{41}{2^3 \cdot 3^2} \\
&= \frac{41}{72}
\end{aligned}$$

The method for finding the lcd for rational expressions is analogous to the process illustrated in Example 2. The only difference is that we use factorizations of polynomials instead of integers.

EXAMPLE 3 **Sums and differences of rational expressions**

Perform the operations and simplify:

$$\frac{6}{x(3x-2)} + \frac{5}{3x-2} - \frac{2}{x^2}$$

SOLUTION The denominators are already in factored form. The lcd is $x^2(3x - 2)$. To obtain three quotients having the denominator $x^2(3x - 2)$, we multiply the numerator and denominator of the first quotient by x, those of the second by x^2, and those of the third by $3x - 2$, which gives us

$$\begin{aligned}
\frac{6}{x(3x-2)} + \frac{5}{3x-2} - \frac{2}{x^2} &= \frac{6}{x(3x-2)} \cdot \frac{x}{x} + \frac{5}{3x-2} \cdot \frac{x^2}{x^2} - \frac{2}{x^2} \cdot \frac{3x-2}{3x-2} \\
&= \frac{6x}{x^2(3x-2)} + \frac{5x^2}{x^2(3x-2)} - \frac{2(3x-2)}{x^2(3x-2)} \\
&= \frac{6x + 5x^2 - 2(3x-2)}{x^2(3x-2)} \\
&= \frac{5x^2 + 4}{x^2(3x-2)}.
\end{aligned}$$

EXAMPLE 4 **Simplifying sums of rational expressions**

Perform the operations and simplify:

$$\frac{2x+5}{x^2+6x+9}+\frac{x}{x^2-9}+\frac{1}{x-3}$$

SOLUTION We begin by factoring denominators:

$$\frac{2x+5}{x^2+6x+9}+\frac{x}{x^2-9}+\frac{1}{x-3}=\frac{2x+5}{(x+3)^2}+\frac{x}{(x+3)(x-3)}+\frac{1}{x-3}$$

Since the lcd is $(x+3)^2(x-3)$, we multiply the numerator and denominator of the first quotient by $x-3$, those of the second by $x+3$, and those of the third by $(x+3)^2$ and then add:

$$\frac{(2x+5)(x-3)}{(x+3)^2(x-3)}+\frac{x(x+3)}{(x+3)^2(x-3)}+\frac{(x+3)^2}{(x+3)^2(x-3)}$$

$$=\frac{(2x^2-x-15)+(x^2+3x)+(x^2+6x+9)}{(x+3)^2(x-3)}$$

$$=\frac{4x^2+8x-6}{(x+3)^2(x-3)}=\frac{2(2x^2+4x-3)}{(x+3)^2(x-3)}$$

A **complex fraction** is a quotient in which the numerator and/or the denominator is a fractional expression. Certain problems in calculus require simplifying complex fractions of the type given in the next example.

EXAMPLE 5 **Simplifying a complex fraction**

Simplify the complex fraction:

$$\frac{\dfrac{2}{x+3}-\dfrac{2}{a+3}}{x-a}$$

SOLUTION We change the numerator of the given expression into a single quotient and then use a property for simplifying quotients:

$$\frac{\dfrac{2}{x+3}-\dfrac{2}{a+3}}{x-a}=\frac{\dfrac{2(a+3)-2(x+3)}{(x+3)(a+3)}}{x-a} \qquad \text{combine fractions in the numerator}$$

$$=\frac{2a-2x}{(x+3)(a+3)}\cdot\frac{1}{x-a} \qquad \text{simplify; property of quotients}$$

$$=\frac{2(a-x)}{(x+3)(a+3)(x-a)} \qquad \text{factor } 2a-2x\text{; property of quotients}$$

$$\overset{\text{if } x\neq a}{\downarrow}$$

$$=-\frac{2}{(x+3)(a+3)} \qquad \text{replace } \frac{a-x}{x-a} \text{ with } -1$$

An alternative method is to multiply the numerator and denominator of the given expression by $(x + 3)(a + 3)$, the lcd of the numerator and denominator, and then simplify the result.

Some quotients that are not rational expressions contain denominators of the form $a + \sqrt{b}$ or $\sqrt{a} + \sqrt{b}$; as in the next example, these quotients can be simplified by multiplying the numerator and denominator by the **conjugate** $a - \sqrt{b}$ or $\sqrt{a} - \sqrt{b}$, respectively. Of course, if $a - \sqrt{b}$ appears, multiply by $a + \sqrt{b}$ instead.

EXAMPLE 6 Rationalizing a denominator

Rationalize the denominator:

$$\frac{1}{\sqrt{x} + \sqrt{y}}$$

SOLUTION

$$\begin{aligned}
\frac{1}{\sqrt{x} + \sqrt{y}} &= \frac{1}{\sqrt{x} + \sqrt{y}} \cdot \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} - \sqrt{y}} && \text{multiply numerator and denominator by the conjugate of } \sqrt{x} + \sqrt{y} \\
&= \frac{\sqrt{x} - \sqrt{y}}{(\sqrt{x})^2 - (\sqrt{y})^2} && \text{property of quotients and difference of squares} \\
&= \frac{\sqrt{x} - \sqrt{y}}{x - y} && \text{law of radicals}
\end{aligned}$$

In calculus it is sometimes necessary to rationalize the *numerator* of a quotient, as shown in the following example.

EXAMPLE 7 Rationalizing a numerator

If $h \neq 0$, rationalize the numerator of

$$\frac{\sqrt{x + h} - \sqrt{x}}{h}.$$

SOLUTION

$$\begin{aligned}
\frac{\sqrt{x + h} - \sqrt{x}}{h} &= \frac{\sqrt{x + h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}} && \text{multiply numerator and denominator by the conjugate of } \sqrt{x + h} - \sqrt{x} \\
&= \frac{(\sqrt{x + h})^2 - (\sqrt{x})^2}{h(\sqrt{x + h} + \sqrt{x})} && \text{property of quotients and difference of squares} \\
&= \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})} && \text{law of radicals} \\
&= \frac{\cancel{h}}{\cancel{h}(\sqrt{x + h} + \sqrt{x})} && \text{simplify} \\
&= \frac{1}{\sqrt{x + h} + \sqrt{x}} && \text{cancel } h \neq 0
\end{aligned}$$

(continued)

It may seem as though we have accomplished very little, since radicals occur in the denominator. In calculus, however, it is of interest to determine what is true if h is very close to zero. Note that if we use the *given* expression we obtain the following:

$$\text{If}\quad h \approx 0, \quad \text{then}\quad \frac{\sqrt{x+h}-\sqrt{x}}{h} \approx \frac{\sqrt{x+0}-\sqrt{x}}{0} = \frac{0}{0},$$

a meaningless expression. If we use the *rationalized* form, however, we obtain the following information:

$$\text{If}\quad h \approx 0, \quad \text{then}\quad \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$\approx \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Certain problems in calculus require simplifying expressions of the type given in the next example.

EXAMPLE 8 **Simplifying a fractional expression**

Simplify, if $h \neq 0$:

$$\frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$$

SOLUTION

$$\frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h} = \frac{\dfrac{x^2-(x+h)^2}{(x+h)^2x^2}}{h} \qquad \text{combine quotients in numerator}$$

$$= \frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2}\cdot\frac{1}{h} \qquad \text{square } x+h\text{; property of quotients}$$

$$= \frac{x^2-x^2-2xh-h^2}{(x+h)^2x^2h} \qquad \text{remove parentheses}$$

$$= \frac{-\cancel{h}(2x+h)}{(x+h)^2x^2\cancel{h}} \qquad \text{simplify; factor out } -h$$

$$= -\frac{2x+h}{(x+h)^2x^2} \qquad \text{cancel } h \neq 0$$

Problems of the type given in the next example also occur in calculus.

EXAMPLE 9 **Simplifying a fractional expression**

Simplify:

$$\frac{3x^2(2x+5)^{1/2} - x^3\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2)}{[(2x+5)^{1/2}]^2}$$

SOLUTION One way to simplify the expression is as follows:

$$\frac{3x^2(2x+5)^{1/2} - x^3\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2)}{[(2x+5)^{1/2}]^2}$$

$$= \frac{3x^2(2x+5)^{1/2} - \dfrac{x^3}{(2x+5)^{1/2}}}{2x+5} \qquad \text{definition of negative exponents}$$

$$= \frac{\dfrac{3x^2(2x+5) - x^3}{(2x+5)^{1/2}}}{2x+5} \qquad \text{combine terms in numerator}$$

$$= \frac{6x^3 + 15x^2 - x^3}{(2x+5)^{1/2}} \cdot \frac{1}{2x+5} \qquad \text{property of quotients}$$

$$= \frac{5x^3 + 15x^2}{(2x+5)^{3/2}} \qquad \text{simplify}$$

$$= \frac{5x^2(x+3)}{(2x+5)^{3/2}} \qquad \text{factor numerator}$$

An alternative simplification is to eliminate the negative power, $-\frac{1}{2}$, in the given expression, as follows:

$$\frac{3x^2(2x+5)^{1/2} - x^3\left(\frac{1}{2}\right)(2x+5)^{-1/2}(2)}{[(2x+5)^{1/2}]^2} \cdot \frac{(2x+5)^{1/2}}{(2x+5)^{1/2}} \qquad \text{multiply numerator and denominator by } (2x+5)^{1/2}$$

$$= \frac{3x^2(2x+5) - x^3}{(2x+5)(2x+5)^{1/2}} \qquad \text{property of quotients and law of exponents}$$

The remainder of the simplification is similar.

A third method of simplification is to first factor out the gcf. In this case, the common factors are x and $(2x + 5)$, and the smallest exponents are 2 and $-\frac{1}{2}$, respectively. Thus, the gcf is $x^2(2x + 5)^{-1/2}$, and we factor the numerator and simplify as follows:

$$\frac{x^2(2x+5)^{-1/2}[3(2x+5)^1 - x]}{(2x+5)^1} = \frac{x^2(5x+15)}{(2x+5)^{3/2}} = \frac{5x^2(x+3)}{(2x+5)^{3/2}}$$

One of the problems in calculus is determining the values of x that make the numerator equal to zero. The simplified form helps us answer this question with relative ease—the values are 0 and -3.

1.4 Exercises

Exer. 1–4: Write the expression as a simplified rational number.

1 $\dfrac{3}{50} + \dfrac{7}{30}$

2 $\dfrac{4}{63} + \dfrac{5}{42}$

3 $\dfrac{5}{24} - \dfrac{3}{20}$

4 $\dfrac{11}{54} - \dfrac{7}{72}$

Exer. 5–48: Simplify the expression.

5 $\dfrac{2x^2 + 7x + 3}{2x^2 - 7x - 4}$

6 $\dfrac{2x^2 + 9x - 5}{3x^2 + 17x + 10}$

7 $\dfrac{y^2 - 25}{y^3 - 125}$

8 $\dfrac{y^2 - 9}{y^3 + 27}$

9 $\dfrac{12 + r - r^2}{r^3 + 3r^2}$

10 $\dfrac{10 + 3r - r^2}{r^4 + 2r^3}$

11 $\dfrac{9x^2 - 4}{3x^2 - 5x + 2} \cdot \dfrac{9x^4 - 6x^3 + 4x^2}{27x^4 + 8x}$

12 $\dfrac{4x^2 - 9}{2x^2 + 7x + 6} \cdot \dfrac{4x^4 + 6x^3 + 9x^2}{8x^7 - 27x^4}$

13 $\dfrac{5a^2 + 12a + 4}{a^4 - 16} \div \dfrac{25a^2 + 20a + 4}{a^2 - 2a}$

14 $\dfrac{a^3 - 8}{a^2 - 4} \div \dfrac{a}{a^3 + 8}$

15 $\dfrac{6}{x^2 - 4} - \dfrac{3x}{x^2 - 4}$

16 $\dfrac{15}{x^2 - 9} - \dfrac{5x}{x^2 - 9}$

17 $\dfrac{2}{3s + 1} - \dfrac{9}{(3s + 1)^2}$

18 $\dfrac{4}{(5s - 2)^2} + \dfrac{s}{5s - 2}$

19 $\dfrac{2}{x} + \dfrac{3x + 1}{x^2} - \dfrac{x - 2}{x^3}$

20 $\dfrac{5}{x} - \dfrac{2x - 1}{x^2} + \dfrac{x + 5}{x^3}$

21 $\dfrac{3t}{t + 2} + \dfrac{5t}{t - 2} - \dfrac{40}{t^2 - 4}$

22 $\dfrac{t}{t + 3} + \dfrac{4t}{t - 3} - \dfrac{18}{t^2 - 9}$

23 $\dfrac{4x}{3x - 4} + \dfrac{8}{3x^2 - 4x} + \dfrac{2}{x}$

24 $\dfrac{12x}{2x + 1} - \dfrac{3}{2x^2 + x} + \dfrac{5}{x}$

25 $\dfrac{2x}{x + 2} - \dfrac{8}{x^2 + 2x} + \dfrac{3}{x}$

26 $\dfrac{5x}{2x + 3} - \dfrac{6}{2x^2 + 3x} + \dfrac{2}{x}$

27 $\dfrac{p^4 + 3p^3 - 8p - 24}{p^3 - 2p^2 - 9p + 18}$

28 $\dfrac{2ac + bc - 6ad - 3bd}{6ac + 2ad + 3bc + bd}$

29 $3 + \dfrac{5}{u} + \dfrac{2u}{3u + 1}$

30 $4 + \dfrac{2}{u} - \dfrac{3u}{u + 5}$

31 $\dfrac{2x + 1}{x^2 + 4x + 4} - \dfrac{6x}{x^2 - 4} + \dfrac{3}{x - 2}$

32 $\dfrac{2x + 6}{x^2 + 6x + 9} + \dfrac{5x}{x^2 - 9} + \dfrac{7}{x - 3}$

33 $\dfrac{\dfrac{b}{a} - \dfrac{a}{b}}{\dfrac{1}{a} - \dfrac{1}{b}}$

34 $\dfrac{\dfrac{1}{x + 2} - 3}{\dfrac{4}{x} - x}$

35 $\dfrac{\dfrac{x}{y^2} - \dfrac{y}{x^2}}{\dfrac{1}{y^2} - \dfrac{1}{x^2}}$

36 $\dfrac{\dfrac{r}{s} + \dfrac{s}{r}}{\dfrac{r^2}{s^2} - \dfrac{s^2}{r^2}}$

37 $\dfrac{y^{-1} + x^{-1}}{(xy)^{-1}}$

38 $\dfrac{y^{-2} - x^{-2}}{y^{-2} + x^{-2}}$

39 $\dfrac{\dfrac{5}{x + 1} + \dfrac{2x}{x + 3}}{\dfrac{x}{x + 1} + \dfrac{7}{x + 3}}$

40 $\dfrac{\dfrac{3}{w} - \dfrac{6}{2w + 1}}{\dfrac{5}{w} + \dfrac{8}{2w + 1}}$

41 $\dfrac{\dfrac{3}{x - 1} - \dfrac{3}{a - 1}}{x - a}$

42 $\dfrac{\dfrac{x + 2}{x} - \dfrac{a + 2}{a}}{x - a}$

43 $\dfrac{(x + h)^2 - 3(x + h) - (x^2 - 3x)}{h}$

44 $\dfrac{(x + h)^3 + 5(x + h) - (x^3 + 5x)}{h}$

45 $\dfrac{\dfrac{1}{(x + h)^3} - \dfrac{1}{x^3}}{h}$

46 $\dfrac{\dfrac{1}{x + h} - \dfrac{1}{x}}{h}$

47 $\dfrac{\dfrac{4}{3x + 3h - 1} - \dfrac{4}{3x - 1}}{h}$

48 $\dfrac{\dfrac{5}{2x + 2h + 3} - \dfrac{5}{2x + 3}}{h}$

Exer. 49–54: Rationalize the denominator.

49 $\dfrac{\sqrt{t} + 5}{\sqrt{t} - 5}$

50 $\dfrac{\sqrt{t} - 4}{\sqrt{t} + 4}$

51 $\dfrac{81x^2 - 16y^2}{3\sqrt{x} - 2\sqrt{y}}$

52 $\dfrac{16x^2 - y^2}{2\sqrt{x} - \sqrt{y}}$

53 $\dfrac{1}{\sqrt[3]{a} - \sqrt[3]{b}}$ (*Hint:* Multiply numerator and denominator by $\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}$.)

54 $\dfrac{1}{\sqrt[3]{x} + \sqrt[3]{y}}$

Exer. 55–60: Rationalize the numerator.

55 $\dfrac{\sqrt{a} - \sqrt{b}}{a^2 - b^2}$

56 $\dfrac{\sqrt{b} + \sqrt{c}}{b^2 - c^2}$

57 $\dfrac{\sqrt{2(x + h) + 1} - \sqrt{2x + 1}}{h}$

58 $\dfrac{\sqrt{x} - \sqrt{x + h}}{h\sqrt{x}\,\sqrt{x + h}}$

59 $\dfrac{\sqrt{1 - x - h} - \sqrt{1 - x}}{h}$

60 $\dfrac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h}$ (*Hint:* Compare with Exercise 53.)

Exer. 61–64: Express as a sum of terms of the form ax^r, where r is a rational number.

61 $\dfrac{4x^2 - x + 5}{x^{2/3}}$

62 $\dfrac{x^2 + 4x - 6}{\sqrt{x}}$

63 $\dfrac{(x^2 + 2)^2}{x^5}$

64 $\dfrac{(\sqrt{x} - 3)^2}{x^3}$

Exer. 65–68: Express as a quotient.

65 $x^{-3} + x^2$

66 $x^{-4} - x$

67 $x^{-1/2} - x^{3/2}$

68 $x^{-2/3} + x^{7/3}$

Exer. 69–82: Simplify the expression.

69 $(2x^2 - 3x + 1)(4)(3x + 2)^3(3) + (3x + 2)^4(4x - 3)$

70 $(6x - 5)^3(2)(x^2 + 4)(2x) + (x^2 + 4)^2(3)(6x - 5)^2(6)$

71 $(x^2 - 4)^{1/2}(3)(2x + 1)^2(2) + (2x + 1)^3\left(\frac{1}{2}\right)(x^2 - 4)^{-1/2}(2x)$

72 $(3x + 2)^{1/3}(2)(4x - 5)(4) + (4x - 5)^2\left(\frac{1}{3}\right)(3x + 2)^{-2/3}(3)$

73 $(3x + 1)^6\left(\frac{1}{2}\right)(2x - 5)^{-1/2}(2) + (2x - 5)^{1/2}(6)(3x + 1)^5(3)$

74 $(x^2 + 9)^4\left(-\frac{1}{3}\right)(x + 6)^{-4/3} + (x + 6)^{-1/3}(4)(x^2 + 9)^3(2x)$

75 $\dfrac{(6x + 1)^3(27x^2 + 2) - (9x^3 + 2x)(3)(6x + 1)^2(6)}{(6x + 1)^6}$

76 $\dfrac{(x^2 - 1)^4(2x) - x^2(4)(x^2 - 1)^3(2x)}{(x^2 - 1)^8}$

77 $\dfrac{(x^2 + 2)^3(2x) - x^2(3)(x^2 + 2)^2(2x)}{[(x^2 + 2)^3]^2}$

78 $\dfrac{(x^2 - 5)^4(3x^2) - x^3(4)(x^2 - 5)^3(2x)}{[(x^2 - 5)^4]^2}$

79 $\dfrac{(x^2 + 4)^{1/3}(3) - (3x)\left(\frac{1}{3}\right)(x^2 + 4)^{-2/3}(2x)}{[(x^2 + 4)^{1/3}]^2}$

80 $\dfrac{(1 - x^2)^{1/2}(2x) - x^2\left(\frac{1}{2}\right)(1 - x^2)^{-1/2}(-2x)}{[(1 - x^2)^{1/2}]^2}$

81 $\dfrac{(4x^2 + 9)^{1/2}(2) - (2x + 3)\left(\frac{1}{2}\right)(4x^2 + 9)^{-1/2}(8x)}{[(4x^2 + 9)^{1/2}]^2}$

82 $\dfrac{(3x + 2)^{1/2}\left(\frac{1}{3}\right)(2x + 3)^{-2/3}(2) - (2x + 3)^{1/3}\left(\frac{1}{2}\right)(3x + 2)^{-1/2}(3)}{[(3x + 2)^{1/2}]^2}$

CHAPTER 1 REVIEW EXERCISES

1 Express as a simplified rational number:

(a) $\left(\frac{2}{3}\right)\left(-\frac{5}{8}\right)$ (b) $\frac{3}{4} + \frac{6}{5}$ (c) $\frac{5}{8} - \frac{6}{7}$ (d) $\frac{3}{4} \div \frac{6}{5}$

2 Replace the symbol □ with either <, >, or = to make the resulting statement true.

(a) $-0.1 \;\square\; -0.001$ (b) $\sqrt{9} \;\square\; -3$

(c) $\frac{1}{6} \;\square\; 0.166$

3 Express the statement as an inequality.

(a) x is negative.

(b) a is between $\frac{1}{2}$ and $\frac{1}{3}$.

(c) The absolute value of x is not greater than 4.

4 Rewrite without using the absolute value symbol, and simplify:

(a) $|-7|$ (b) $\dfrac{|-5|}{-5}$ (c) $|3^{-1} - 2^{-1}|$

5 If points A, B, and C on a coordinate line have coordinates -8, 4, and -3, respectively, find the distance:

(a) $d(A, C)$ (b) $d(C, A)$ (c) $d(B, C)$

6 Express the indicated statement as an inequality involving the absolute value symbol.

(a) $d(x, -2)$ is at least 7.

(b) $d(4, x)$ is less than 4.

Exer. 7–8: Rewrite the expression without using the absolute value symbol, and simplify the result.

7 $|x + 3|$ if $x \leq -3$

8 $|(x - 2)(x - 3)|$ if $2 < x < 3$

9 Determine whether the expression is true for all values of the variables, whenever the expression is defined.

(a) $(x + y)^2 = x^2 + y^2$ (b) $\dfrac{1}{\sqrt{x + y}} = \dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}}$

(c) $\dfrac{1}{\sqrt{c} - \sqrt{d}} = \dfrac{\sqrt{c} + \sqrt{d}}{c - d}$

10 Express the number in scientific form.

(a) 93,700,000,000 (b) 0.000 004 02

11 Express the number in decimal form.

(a) 6.8×10^7 (b) 7.3×10^{-4}

12 (a) Approximate $|\sqrt{5} - 17^2|$ to four decimal places.

(b) Express the answer in part (a) in scientific notation accurate to four significant figures.

Exer. 13–14: Express the number in the form a/b, where a and b are integers.

13 $-3^2 + 2^0 + 27^{-2/3}$

14 $\left(\frac{1}{2}\right)^0 - 1^2 + 16^{-3/4}$

Exer. 15–40: Simplify the expression, and rationalize the denominator when appropriate.

15 $(3a^2b)^2(2ab^3)$

16 $\dfrac{6r^3y^2}{2r^5y}$

17 $\dfrac{(3x^2y^{-3})^{-2}}{x^{-5}y}$

18 $\left(\dfrac{a^{2/3}b^{3/2}}{a^2b}\right)^6$

19 $(-2p^2q)^3\left(\dfrac{p}{4q^2}\right)^2$

20 $c^{-4/3}c^{3/2}c^{1/6}$

21 $\left(\dfrac{xy^{-1}}{\sqrt{z}}\right)^4 \div \left(\dfrac{x^{1/3}y^2}{z}\right)^3$

22 $\left(\dfrac{-64x^3}{z^6y^9}\right)^{2/3}$

23 $[(a^{2/3}b^{-2})^3]^{-1}$

24 $\dfrac{(3u^2v^5w^{-4})^3}{(2uv^{-3}w^2)^4}$

25 $\dfrac{r^{-1} + s^{-1}}{(rs)^{-1}}$

26 $(u + v)^3(u + v)^{-2}$

27 $s^{5/2}s^{-4/3}s^{-1/6}$

28 $x^{-2} - y^{-1}$

29 $\sqrt[3]{(x^4y^{-1})^6}$

30 $\sqrt[3]{8x^5y^3z^4}$

31 $\dfrac{1}{\sqrt[3]{4}}$

32 $\sqrt{\dfrac{a^2b^3}{c}}$

33 $\sqrt[3]{4x^2y}\ \sqrt[3]{2x^5y^2}$

34 $\sqrt[4]{(-4a^3b^2c)^2}$

35 $\dfrac{1}{\sqrt{t}}\left(\dfrac{1}{\sqrt{t}} - 1\right)$

36 $\sqrt{\sqrt[3]{(c^3d^6)^4}}$

37 $\dfrac{\sqrt{12x^4y}}{\sqrt{3x^2y^5}}$

38 $\sqrt[3]{(a + 2b)^3}$

39 $\sqrt[3]{\dfrac{1}{2\pi^2}}$

40 $\sqrt[3]{\dfrac{x^2}{9y}}$

Exer. 41–44: Rationalize the denominator.

41 $\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

42 $\dfrac{1}{\sqrt{a} + \sqrt{a - 2}}$

43 $\dfrac{81x^2 - y^2}{3\sqrt{x} + \sqrt{y}}$

44 $\dfrac{3 + \sqrt{x}}{3 - \sqrt{x}}$

Exer. 45–62: Express as a polynomial.

45 $(3x^3 - 4x^2 + x - 7) + (x^4 - 2x^3 + 3x^2 + 5)$

46 $(4z^4 - 3z^2 + 1) - z(z^3 + 4z^2 - 4)$

47 $(x + 4)(x + 3) - (2x - 1)(x - 5)$

48 $(4x - 5)(2x^2 + 3x - 7)$

49 $(3y^3 - 2y^2 + y + 4)(y^2 - 3)$

50 $(3x + 2)(x - 5)(5x + 4)$

51 $(a - b)(a^3 + a^2b + ab^2 + b^3)$

52 $\dfrac{9p^4q^3 - 6p^2q^4 + 5p^3q^2}{3p^2q^2}$

53 $(3a - 5b)(2a + 7b)$

54 $(4r^2 - 3s)^2$

55 $(13a^2 + 4b)(13a^2 - 4b)$

56 $(a^3 - a^2)^2$

57 $(3y + x)^2$

58 $(c^2 - d^2)^3$

59 $(2a + b)^3$

60 $(x^2 - 2x + 3)^2$

61 $(3x + 2y)^2(3x - 2y)^2$

62 $(a + b + c + d)^2$

Exer. 63–78: Factor the polynomial.

63 $60xw + 70w$

64 $2r^4s^3 - 8r^2s^5$

65 $28x^2 + 4x - 9$

66 $16a^4 + 24a^2b^2 + 9b^4$

67 $2wy + 3yx - 8wz - 12zx$

68 $2c^3 - 12c^2 + 3c - 18$

69 $8x^3 + 64y^3$

70 $u^3v^4 - u^6v$

71 $p^8 - q^8$

72 $x^4 - 8x^3 + 16x^2$

73 $w^6 + 1$

74 $3x + 6$

75 $x^2 + 36$

76 $x^2 - 49y^2 - 14x + 49$

77 $x^5 - 4x^3 + 8x^2 - 32$

78 $4x^4 + 12x^3 + 20x^2$

Exer. 79–90: Simplify the expression.

79 $\dfrac{6x^2 - 7x - 5}{4x^2 + 4x + 1}$

80 $\dfrac{r^3 - t^3}{r^2 - t^2}$

81 $\dfrac{6x^2 - 5x - 6}{x^2 - 4} \div \dfrac{2x^2 - 3x}{x + 2}$

82 $\dfrac{2}{4x - 5} - \dfrac{5}{10x + 1}$

83 $\dfrac{7}{x + 2} + \dfrac{3x}{(x + 2)^2} - \dfrac{5}{x}$

84 $\dfrac{x + x^{-2}}{1 + x^{-2}}$

85 $\dfrac{1}{x} - \dfrac{2}{x^2 + x} - \dfrac{3}{x + 3}$

86 $(a^{-1} + b^{-1})^{-1}$

87 $\dfrac{x + 2 - \dfrac{3}{x + 4}}{\dfrac{x}{x + 4} + \dfrac{1}{x + 4}}$

88 $\dfrac{\dfrac{x}{x + 2} - \dfrac{4}{x + 2}}{x - 3 - \dfrac{6}{x + 2}}$

89 $(x^2 + 1)^{3/2}(4)(x + 5)^3 + (x + 5)^4\left(\frac{3}{2}\right)(x^2 + 1)^{1/2}(2x)$

90 $\dfrac{(4 - x^2)\left(\frac{1}{3}\right)(6x + 1)^{-2/3}(6) - (6x + 1)^{1/3}(-2x)}{(4 - x^2)^2}$

91 Express $\dfrac{(x + 5)^2}{\sqrt{x}}$ as a sum of terms of the form ax^r, where r is a rational number.

92 Express $x^3 + x^{-1}$ as a quotient.

93 **Red blood cells in a body** The body of an average person contains 5.5 liters of blood and about 5 million red blood cells per cubic millimeter of blood. Given that $1\text{ L} = 10^6\text{ mm}^3$, estimate the number of red blood cells in an average person's body.

94 **Heartbeats in a lifetime** A healthy heart beats 70 to 90 times per minute. Estimate the number of heartbeats in the lifetime of an individual who lives to age 80.

95 **Body surface area** At age 2 years, a typical boy is 91.2 centimeters tall and weighs 13.7 kilograms. Use the DuBois and DuBois formula, $S = (0.007184)w^{0.425}h^{0.725}$, where w is weight and h is height, to find the body surface area S (in square meters).

96 **Adiabatic expansion** A gas is said to expand *adiabatically* if there is no loss or gain of heat. The formula for the adiabatic expansion of air is $pv^{-1.4} = c$, where p is the pressure, v is the volume, and c is a constant. If, at a certain instant, the pressure is 40 dyne/cm^2 and the volume is 60 cm^3, find the value of c (a *dyne* is the unit of force in the cgs system).

CHAPTER 1 DISCUSSION EXERCISES

1 **Credit card cash back** For every \$10 charged to a particular credit card, 1 point is awarded. At the end of the year, 100 points can be exchanged for \$1 in cash back. What percent discount does this cash back represent in terms of the amount of money charged to the credit card?

2 Determine the conditions under which $\sqrt{a^2 + b^2} = a + b$.

3 Show that the sum of squares $x^2 + 25$ can be factored by adding and subtracting a particular term and following the method demonstrated in Example 10(c) of Section 1.3.

4 What is the difference between the expressions $\dfrac{1}{x + 1}$ and $\dfrac{x - 1}{x^2 - 1}$?

5 Write the quotient of two arbitrary second-degree polynomials in x, and evaluate the quotient with several large values of x. What general conclusion can you reach about such quotients?

6 Simplify the expression $\dfrac{3x^2 - 5x - 2}{x^2 - 4}$. Now evaluate both expressions with a value of x ($x \neq \pm 2$). Discuss what this evaluation proves (or doesn't) and what your simplification proves (or doesn't).

7 **Party trick** To guess your partner's age and height, have him/her do the following:

1 Write down his/her age.
2 Multiply it by 2.
3 Add 5.
4 Multiply this sum by 50.
5 Subtract 365.
6 Add his/her height (in inches).
7 Add 115.

The first two digits of the result equal his/her age, and the last two digits equal his/her height. Explain why this is true.

8 **Circuits problem** In a particular circuits problem, the output voltage is defined by

$$V_{\text{out}} = I_{\text{in}}\left(-\frac{RXi}{R - Xi}\right),$$

where $I_{\text{in}} = \dfrac{V_{\text{in}}}{Z_{\text{in}}}$ and $Z_{\text{in}} = \dfrac{R^2 - X^2 - 3RXi}{R - Xi}$. Find a formula for V_{out} in terms of V_{in} when R is equal to X.

9 **Relating baseball records** Based on the number of runs scored (S) and runs allowed (A), the Pythagorean winning percentage estimates what a baseball team's winning percentage should be. This formula, developed by baseball statistician Bill James, has the form

$$\frac{S^x}{S^x + A^x}.$$

James determined that $x = 1.83$ yields the most accurate results.

The 1927 New York Yankees are generally regarded as one of the best teams in baseball history. Their record was 110 wins and 44 losses. They scored 975 runs while allowing only 599.

(a) Find their Pythagorean win–loss record.

(b) Estimate the value of x (to the nearest 0.01) that best predicts the 1927 Yankees' actual win–loss record.

Answers to Selected Exercises

A *Student's Solutions Manual* to accompany this textbook is available from your college bookstore. The guide contains detailed solutions to approximately one-half of the exercises, as well as strategies for solving other exercises in the text.

Chapter 1

EXERCISES 1.1

1 **(a)** Negative **(b)** Positive **(c)** Negative **(d)** Positive
3 **(a)** $<$ **(b)** $>$ **(c)** $=$
5 **(a)** $>$ **(b)** $>$ **(c)** $>$
7 **(a)** $x < 0$ **(b)** $y \geq 0$ **(c)** $q \leq \pi$ **(d)** $2 < d < 4$ **(e)** $t \geq 5$ **(f)** $-z \leq 3$ **(g)** $\dfrac{p}{q} \leq 7$ **(h)** $\dfrac{1}{w} \geq 9$ **(i)** $|x| > 7$
9 **(a)** 5 **(b)** 3 **(c)** 11
11 **(a)** -15 **(b)** -3 **(c)** 11
13 **(a)** $4 - \pi$ **(b)** $4 - \pi$ **(c)** $1.5 - \sqrt{2}$
15 **(a)** 4 **(b)** 12 **(c)** 12 **(d)** 8
17 **(a)** 10 **(b)** 9 **(c)** 9 **(d)** 19
19 $|7 - x| < 5$
21 $|-3 - x| \geq 8$
23 $|x - 4| \leq 3$
25 $-x - 3$
27 $2 - x$
29 $b - a$
31 $x^2 + 4$
33 $\neq$
35 $=$
37 $\neq$
39 $=$
41 **(a)** 8.4652 **(b)** 14.1428
43 **(a)** 6.557×10^{-1} **(b)** 6.708×10^{1}
45 Construct a right triangle with sides of lengths $\sqrt{2}$ and 1. The hypotenuse will have length $\sqrt{3}$. Next, construct a right triangle with sides of lengths $\sqrt{3}$ and $\sqrt{2}$. The hypotenuse will have length $\sqrt{5}$.
47 The large rectangle has area $a(b + c)$. The sum of the areas of the two small rectangles is $ab + ac$.
49 **(a)** 4.27×10^5 **(b)** 9.8×10^{-8} **(c)** 8.1×10^8
51 **(a)** 830,000 **(b)** 0.000 000 000 002 9 **(c)** 563,000,000
53 1.7×10^{-24}
55 5.87×10^{12}
57 1.678×10^{-24} g
59 4.1472×10^6 frames
61 **(a)** 201.6 lb **(b)** 32.256 tons

EXERCISES 1.2

1 $\dfrac{16}{81}$ **3** $\dfrac{9}{8}$ **5** $\dfrac{-47}{3}$ **7** $\dfrac{1}{8}$ **9** $\dfrac{1}{25}$ **11** $8x^9$
13 $\dfrac{6}{x}$ **15** $-2a^{14}$ **17** $\dfrac{9}{2}$ **19** $\dfrac{12u^{11}}{v^2}$ **21** $\dfrac{4}{xy}$
23 $\dfrac{9y^6}{x^8}$ **25** $\dfrac{81}{64}y^6$ **27** $\dfrac{s^6}{4r^8}$ **29** $\dfrac{20y}{x^3}$ **31** $9x^{10}y^{14}$
33 $8a^2$ **35** $24x^{3/2}$ **37** $\dfrac{1}{9a^4}$ **39** $\dfrac{8}{x^{1/2}}$ **41** $4x^2y^4$
43 $\dfrac{3}{x^3y^2}$ **45** 1 **47** $x^{3/4}$ **49** $(a + b)^{2/3}$
51 $(x^2 + y^2)^{1/2}$ **53** **(a)** $4x\sqrt{x}$ **(b)** $8x\sqrt{x}$
55 **(a)** $8 - \sqrt[3]{y}$ **(b)** $\sqrt[3]{8 - y}$ **57** 9 **59** $-2\sqrt[5]{2}$
61 $\dfrac{1}{2}\sqrt[3]{4}$ **63** $\dfrac{3y^3}{x^2}$ **65** $\dfrac{2a^2}{b}$ **67** $\dfrac{1}{2y^2}\sqrt{6xy}$
69 $\dfrac{xy}{3}\sqrt[3]{6y}$ **71** $\dfrac{x}{3}\sqrt[4]{15x^2y^3}$ **73** $\dfrac{1}{2}\sqrt[5]{20x^4y^2}$
75 $\dfrac{3x^5}{y^2}$ **77** $\dfrac{2x}{y^2}\sqrt[5]{x^2y^4}$ **79** $-3tv^2$ **81** $|x^3|y^2$
83 $x^2|y - 1|^3$ **85** $\neq$; $(a^r)^2 = a^{2r} \neq a^{(r^2)}$
87 $\neq$; $(ab)^{xy} = a^{xy}b^{xy} \neq a^xb^y$
89 $=$; $\sqrt[n]{\dfrac{1}{c}} = \left(\dfrac{1}{c}\right)^{1/n} = \dfrac{1^{1/n}}{c^{1/n}} = \dfrac{1}{\sqrt[n]{c}}$
91 **(a)** 1.5518 **(b)** 8.5499
93 **(a)** 2.0351 **(b)** 3.9670 **95** \$232,825.78
97 2.82 m **99** The 120-kg lifter
101

Height	Weight	Height	Weight
64	137	72	168
65	141	73	172
66	145	74	176
67	148	75	180
68	152	76	184
69	156	77	188
70	160	78	192
71	164	79	196

EXERCISES 1.3

1 $12x^3 - 13x + 1$ **3** $x^3 - 2x^2 + 4$
5 $6x^2 + x - 35$ **7** $15x^2 + 31xy + 14y^2$
9 $6u^2 - 13u - 12$ **11** $6x^3 + 37x^2 + 30x - 25$
13 $3t^4 + 5t^3 - 15t^2 + 9t - 10$
15 $2x^6 + 2x^5 - 2x^4 + 8x^3 + 10x^2 - 10x - 10$
17 $4y^2 - 5x$ **19** $3v^2 - 2u^2 + uv^2$ **21** $4x^2 - 9y^2$
23 $x^4 - 4y^2$ **25** $x^4 + 5x^2 - 36$
27 $9x^2 + 12xy + 4y^2$ **29** $x^4 - 6x^2y^2 + 9y^4$
31 $x^4 - 8x^2 + 16$ **33** $x - y$ **35** $x - y$
37 $x^3 - 6x^2y + 12xy^2 - 8y^3$
39 $8x^3 + 36x^2y + 54xy^2 + 27y^3$
41 $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$
43 $4x^2 + y^2 + 9z^2 + 4xy - 12xz - 6yz$

45 $s(r + 4t)$ **47** $3a^2b(b - 2)$ **49** $3x^2y^2(y - 3x)$
51 $5x^3y^2(3y^3 - 5x + 2x^3y^2)$ **53** $(8x + 3)(x - 7)$
55 Irreducible **57** $(3x - 4)(2x + 5)$
59 $(3x - 5)(4x - 3)$ **61** $(2x - 5)^2$ **63** $(5z + 3)^2$
65 $(5x + 2y)(9x + 4y)$ **67** $(6r + 5t)(6r - 5t)$
69 $(z^2 + 8w)(z^2 - 8w)$ **71** $x^2(x + 2)(x - 2)$
73 Irreducible **75** $3(5x + 4y)(5x - 4y)$
77 $(4x + 3)(16x^2 - 12x + 9)$
79 $(4x - y^2)(16x^2 + 4xy^2 + y^4)$
81 $(7x + y^3)(49x^2 - 7xy^3 + y^6)$
83 $(5 - 3x)(25 + 15x + 9x^2)$
85 $(2x + y)(a - 3b)$ **87** $3(x + 3)(x - 3)(x + 1)$
89 $(x - 1)(x + 2)(x^2 + x + 1)$ **91** $(a^2 + b^2)(a - b)$
93 $(a + b)(a - b)(a^2 - ab + b^2)(a^2 + ab + b^2)$
95 $(x + 2 + 3y)(x + 2 - 3y)$
97 $(y + 4 + x)(y + 4 - x)$
99 $(y + 2)(y^2 - 2y + 4)(y - 1)(y^2 + y + 1)$
101 $(x^8 + 1)(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$
103 Area of I is $(x - y)x$, area of II is $(x - y)y$, and
$$A = x^2 - y^2 = (x - y)x + (x - y)y = (x - y)(x + y).$$
105 **(a)** 1525.7; 1454.7
(b) As people age, they require fewer calories. Coefficients of w and h are positive because large people require more calories.

EXERCISES 1.4

1 $\dfrac{22}{75}$ **3** $\dfrac{7}{120}$ **5** $\dfrac{x + 3}{x - 4}$ **7** $\dfrac{y + 5}{y^2 + 5y + 25}$
9 $\dfrac{4 - r}{r^2}$ **11** $\dfrac{x}{x - 1}$ **13** $\dfrac{a}{(a^2 + 4)(5a + 2)}$
15 $\dfrac{-3}{x + 2}$ **17** $\dfrac{6s - 7}{(3s + 1)^2}$ **19** $\dfrac{5x^2 + 2}{x^3}$
21 $\dfrac{4(2t + 5)}{t + 2}$ **23** $\dfrac{2(2x + 3)}{3x - 4}$ **25** $\dfrac{2x - 1}{x}$
27 $\dfrac{p^2 + 2p + 4}{p - 3}$ **29** $\dfrac{11u^2 + 18u + 5}{u(3u + 1)}$ **31** $-\dfrac{x + 5}{(x + 2)^2}$
33 $a + b$ **35** $\dfrac{x^2 + xy + y^2}{x + y}$ **37** $x + y$
39 $\dfrac{2x^2 + 7x + 15}{x^2 + 10x + 7}$ **41** $-\dfrac{3}{(x - 1)(a - 1)}$
43 $2x + h - 3$ **45** $-\dfrac{3x^2 + 3xh + h^2}{x^3(x + h)^3}$
47 $\dfrac{-12}{(3x + 3h - 1)(3x - 1)}$ **49** $\dfrac{t + 10\sqrt{t} + 25}{t - 25}$
51 $(9x + 4y)\left(3\sqrt{x} + 2\sqrt{y}\right)$ **53** $\dfrac{\sqrt[3]{a^2} + \sqrt[3]{ab} + \sqrt[3]{b^2}}{a - b}$
55 $\dfrac{1}{(a + b)\left(\sqrt{a} + \sqrt{b}\right)}$ **57** $\dfrac{2}{\sqrt{2(x + h) + 1} + \sqrt{2x + 1}}$
59 $\dfrac{-1}{\sqrt{1 - x - h} + \sqrt{1 - x}}$ **61** $4x^{4/3} - x^{1/3} + 5x^{-2/3}$
63 $x^{-1} + 4x^{-3} + 4x^{-5}$ **65** $\dfrac{1 + x^5}{x^3}$ **67** $\dfrac{1 - x^2}{x^{1/2}}$
69 $(3x + 2)^3(36x^2 - 37x + 6)$
71 $\dfrac{(2x + 1)^2(8x^2 + x - 24)}{(x^2 - 4)^{1/2}}$ **73** $\dfrac{(3x + 1)^5(39x - 89)}{(2x - 5)^{1/2}}$
75 $\dfrac{27x^2 - 24x + 2}{(6x + 1)^4}$ **77** $\dfrac{4x(1 - x^2)}{(x^2 + 2)^4}$ **79** $\dfrac{x^2 + 12}{(x^2 + 4)^{4/3}}$
81 $\dfrac{6(3 - 2x)}{(4x^2 + 9)^{3/2}}$

CHAPTER 1 REVIEW EXERCISES

1 **(a)** $-\dfrac{5}{12}$ **(b)** $\dfrac{39}{20}$ **(c)** $-\dfrac{13}{56}$ **(d)** $\dfrac{5}{8}$
2 **(a)** $<$ **(b)** $>$ **(c)** $>$
3 **(a)** $x < 0$ **(b)** $\dfrac{1}{3} < a < \dfrac{1}{2}$ **(c)** $|x| \le 4$
4 **(a)** 7 **(b)** -1 **(c)** $\dfrac{1}{6}$ **5** **(a)** 5 **(b)** 5 **(c)** 7
6 **(a)** $|-2 - x| \ge 7$ **(b)** $|x - 4| < 4$
7 $-x - 3$ **8** $-(x - 2)(x - 3)$
9 **(a)** No **(b)** No **(c)** Yes
10 **(a)** 9.37×10^{10} **(b)** 4.02×10^{-6}
11 **(a)** 68,000,000 **(b)** 0.000 73
12 **(a)** 286.7639 **(b)** 2.868×10^2 **13** $\dfrac{-71}{9}$
14 $\dfrac{1}{8}$ **15** $18a^5b^5$ **16** $\dfrac{3y}{r^2}$ **17** $\dfrac{xy^5}{9}$ **18** $\dfrac{b^3}{a^8}$
19 $-\dfrac{p^8}{2q}$ **20** $c^{1/3}$ **21** $\dfrac{x^3z}{y^{10}}$ **22** $\dfrac{16x^2}{z^4y^6}$ **23** $\dfrac{b^6}{a^2}$
24 $\dfrac{27u^2v^{27}}{16w^{20}}$ **25** $s + r$ **26** $u + v$ **27** s
28 $\dfrac{y - x^2}{x^2y}$ **29** $\dfrac{x^8}{y^2}$ **30** $2xyz\sqrt[3]{x^2z}$ **31** $\dfrac{1}{2}\sqrt[3]{2}$
32 $\dfrac{ab}{c}\sqrt{bc}$ **33** $2x^2y\sqrt[3]{x}$ **34** $2ab\sqrt{ac}$
35 $\dfrac{1 - \sqrt{t}}{t}$ **36** c^2d^4 **37** $\dfrac{2x}{y^2}$ **38** $a + 2b$
39 $\dfrac{1}{2\pi}\sqrt[3]{4\pi}$ **40** $\dfrac{1}{3y}\sqrt[3]{3x^2y^2}$
41 $\dfrac{1 - 2\sqrt{x} + x}{1 - x}$ **42** $\dfrac{\sqrt{a} - \sqrt{a - 2}}{2}$
43 $(9x + y)\left(3\sqrt{x} - \sqrt{y}\right)$ **44** $\dfrac{x + 6\sqrt{x} + 9}{9 - x}$
45 $x^4 + x^3 - x^2 + x - 2$
46 $3z^4 - 4z^3 - 3z^2 + 4z + 1$ **47** $-x^2 + 18x + 7$

48 $8x^3 + 2x^2 - 43x + 35$
49 $3y^5 - 2y^4 - 8y^3 + 10y^2 - 3y - 12$
50 $15x^3 - 53x^2 - 102x - 40$ **51** $a^4 - b^4$
52 $3p^2q - 2q^2 + \dfrac{5}{3}p$ **53** $6a^2 + 11ab - 35b^2$
54 $16r^4 - 24r^2s + 9s^2$ **55** $169a^4 - 16b^2$
56 $a^6 - 2a^5 + a^4$ **57** $9y^2 + 6xy + x^2$
58 $c^6 - 3c^4d^2 + 3c^2d^4 - d^6$ **59** $8a^3 + 12a^2b + 6ab^2 + b^3$
60 $x^4 - 4x^3 + 10x^2 - 12x + 9$ **61** $81x^4 - 72x^2y^2 + 16y^4$
62 $a^2 + b^2 + c^2 + d^2 + 2(ab + ac + ad + bc + bd + cd)$
63 $10w(6x + 7)$ **64** $2r^2s^3(r + 2s)(r - 2s)$
65 $(14x + 9)(2x - 1)$ **66** $(4a^2 + 3b^2)^2$
67 $(y - 4z)(2w + 3x)$ **68** $(2c^2 + 3)(c - 6)$
69 $8(x + 2y)(x^2 - 2xy + 4y^2)$
70 $u^3v(v - u)(v^2 + uv + u^2)$
71 $(p^4 + q^4)(p^2 + q^2)(p + q)(p - q)$ **72** $x^2(x - 4)^2$
73 $(w^2 + 1)(w^4 - w^2 + 1)$ **74** $3(x + 2)$
75 Irreducible **76** $(x - 7 + 7y)(x - 7 - 7y)$
77 $(x - 2)(x + 2)^2(x^2 - 2x + 4)$ **78** $4x^2(x^2 + 3x + 5)$
79 $\dfrac{3x - 5}{2x + 1}$ **80** $\dfrac{r^2 + rt + t^2}{r + t}$ **81** $\dfrac{3x + 2}{x(x - 2)}$
82 $\dfrac{27}{(4x - 5)(10x + 1)}$ **83** $\dfrac{5x^2 - 6x - 20}{x(x + 2)^2}$ **84** $\dfrac{x^3 + 1}{x^2 + 1}$
85 $\dfrac{-2x^2 - x - 3}{x(x + 1)(x + 3)}$ **86** $\dfrac{ab}{a + b}$ **87** $x + 5$
88 $\dfrac{1}{x + 3}$ **89** $(x^2 + 1)^{1/2}(x + 5)^3(7x^2 + 15x + 4)$
90 $\dfrac{2(5x^2 + x + 4)}{(6x + 1)^{2/3}(4 - x^2)^2}$ **91** $x^{3/2} + 10x^{1/2} + 25x^{-1/2}$
92 $\dfrac{x^4 + 1}{x}$ **93** 2.75×10^{13} cells
94 Between 2.94×10^9 and 3.78×10^9 beats
95 $0.58\ \text{m}^2$ **96** 0.13 dyne-cm

This page contains answers for this chapter only.

CHAPTER 1 DISCUSSION EXERCISES

1 0.1% **2** Either $a = 0$ or $b = 0$
3 Add and subtract $10x$; $x + 5 \pm \sqrt{10x}$ are the factors.
4 The first expression can be evaluated at $x = 1$.
5 They get close to the ratio of leading coefficients as x gets larger.
7 If x is the age and y is the height, show that the final value is $100x + y$.
8 $V_{\text{out}} = \frac{1}{3}V_{\text{in}}$ **9** **(a)** 109–45 **(b)** 1.88