MA 15910 Lesson 20 Notes
(Calculus part of text) Section 4.4
Derivatives of Exponential Functions

How does one find the derivative of a natural exponential function, $y = f(x) = e^x$?
I will derive the formula for a derivative of the function $f(x) = e^x$ using the limit definition for a derivative.

$$derivative = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$\frac{d(e^x)}{dx} = \lim_{h \to 0}\left(\frac{e^{x+h} - e^x}{h}\right)$$ (Remember: Exponents are added when powers are multiplied.)

$$= \lim_{h \to 0}\left(\frac{e^x e^h - e^x}{h}\right)$$ Factor out an e^x in the numerator.

$$= \lim_{h \to 0}\left(\frac{e^x(e^h - 1)}{h}\right)$$ Rewrite the fraction as a product.

$$= \lim_{h \to 0}\left[e^x\left(\frac{e^h - 1}{h}\right)\right]$$

$$= \left(\lim_{h \to 0} e^x\right)\left(\lim_{h \to 0}\left(\frac{e^h - 1}{h}\right)\right)$$ For h values very close to zero, $\left(\frac{e^h - 1}{h}\right) \approx 1$

(You can see a table on page 228 of the calculus part of the textbook to verify this.)

Therefore: $\frac{d(e^x)}{dx} = \lim_{h \to 0}(e^x) \cdot 1 = e^x$ $\quad \frac{d(e^x)}{dx} = e^x$

<u>This argument or proof leads to the following conclusion and derivative rule.</u>

Derivative of e^x : $\frac{d}{dx}\left(e^x\right) = e^x$

Note: MA 15910 students are only responsible for studying derivatives of natural exponential functions, those with base *e*. For your information only, I have included here (next page, also in the textbook) the derivative rule for an exponential function of any other base.

Finding Derivatives of exponential functions with bases other than *e*:

OPTIONAL
**To find the rule to determine the derivative of exponential functions with bases other than *e*, we use the following fact. By the 'composition of inverse functions' in algebra, $e^{\ln a} = a$. **

$e^{\ln a} = a$ by the definition of inverse functions

$e^{(\ln a)x} = \left(e^{(\ln a)}\right)^x$ Product rule of exponents

$= a^x$

Therefore: $a^x = e^{(\ln a)x}$

Use this fact to find the derivative of a^x.

Derivative of a^x:

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{(\ln a)x}\right)$$
$$= (e^{(\ln a)x})(\ln a)$$
$$= (\ln a)a^x$$

$$\frac{d}{dx}(a^x) = (\ln a)a^x$$

Note: There are no homework problems assigned with exponential functions with any base other than *e*. Therefore the above derivative rule is not necessary for students to study. I simply provide it for information purposes only.

Remember the 'chain rule' must be used if the exponent is an expression (a function) other than the variable *x*.

Derivatives of exponential functions (base *e*) using the chain rule:

$$\frac{d}{dx}\left(e^{g(x)}\right) = (e^{g(x)})(g'(x))$$

The derivative of $e^{g(x)}$ Is itself times the derivative of the exponent, $g(x)$.

<u>Ex</u> 1: Find the derivatives:

$a)\quad y = e^{3x}$

$b)\quad y = e^{(5x+6x^2)}$

c) $D_x(3.8e^{1.5x}) =$

d) $h(x) = -12e^{x^2}$

Always write answers to these <u>derivatives in factored form</u>. Factor out a GCF.

e) $y = 4e^{4x^2+2x}$

f) $y = 3x^3e^{-4x}$ (Use product rule.)

$1st: 3x^3$ $2nd: e^{-4x}$

g) $g(x) = (5x^2 - 7x^3)e^{-3x}$ (Use product rule.)

$1st: (5x^2 - 7x^3)$ $2nd: e^{-3x}$ Chain rule will be used in 'dee two'.

$h)\quad y = \dfrac{e^{2x}}{3x^2 + 5}$

Rule for finding the derivative of a natural exponential function is used within the quotient rule.

$i)\quad f(x) = (e^{4x^2} + 10x)^4$

Chain rule is used with the 4^{th} power being the outer function and the inside of the parentheses is the inner function.

j) $y = \frac{200}{8+3e^x}$

Rule for finding derivative of an exponential function is used within the quotient rule.

or Rewrite the function as $y = 200(8+3e^x)^{-1}$ and use the chain rule.

Ex 2:
The quantity (in grams) of a radioactive substance present after *t* years is given by the model $Q(t) = 100e^{-0.421t}$.

a) Find the quantity when $t = 0$, $t = 2$, $t = 5$.

b) Find the rate of change after (a) 2 years, (b) 5 years.

<u>Ex 3</u>:
The growth of the world population (in millions) can be approximated by the function $A(t) = 3100e^{(0.0166t)}$ where t is the number of years since 1960. Find the of <u>instantaneous rate of change</u> in the world population for the year (a) 2010 and (b) 2015.

<u>Ex 4</u>:

The amount (in grams) of a sample of an element present after t years is given by $A(t) = 400e^{-0.4t}$. Find the rate of change of the quantity after (a) 3 years, (b) 5 years, (c) 15 years (d) 30 years, and (e) 100 years.