MA 15910 Lesson 21 Notes

How would a person solve an equation with a variable in an exponent, such as $2^{(x-5)} = 9$? (We cannot re-write this equation easily with the same base.) A notation was developed so that equations such as this one could be solved. This 'new' function is called a logarithm or a logarithmic function.

Definition of a logarithm or logarithmic function:
Let $a > 0$, $a \neq 1$, and $x > 0$ (a any positive number other than 1 and x any positive number), then the logarithm y and the logarithmic function is defined as

f(x) = y = log$_a$ x and is equivalent to a^y = x.

When written in logarithmic form $\left(f(x) = \log_a x\right)$, the x is called an **argument**. **The expression $\log_a x$ is called a logarithmic expression. The $f(x)$ or y is a logarithm or exponent.**

It is important to understand that a logarithm is an exponent!

It is important that students are able to convert between 'logarithmic form' and 'exponential form'.

logarithmic form		exponential form
$y = \log_a x$	$\Leftrightarrow$	$a^y = x$
y is a logarithm		y is an exponent
x is the argument (or power)		x is a power (or argument)
a is the base		a is the base

Ex 1: Write each exponential expression as a logarithmic expression. (Convert from exponential form to logarithmic form.)

$a)$ $4^2 = 16$ $b)$ $m^{-5} = q$ $c)$ $\left(\frac{2}{3}\right)^n = \frac{8}{27}$

$d)$ $10^5 = a$ $e)$ $e^{(x-2)} = 8$ $f)$ $n^{12} = 5000$

$g)$ $e^5 = r$ $h)$ $10^2 = 100$

<u>Ex 2</u>: Write each logarithmic expression as an exponential expression. (Convert from logarithmic form to exponential form.)

$a)\ \log_5 125 = 3$ $b)\ \log_r 100 = 3$ $c)\ \log_\pi n = -3$

$d)\ \log_{\left(\frac{1}{4}\right)} 9 = m$ $e)\ \log_b 25 = (q+1)$ $f)\ \log_{0.2} n = -4$

$g)\ \log 1000 = 3$ $h)\ \ln r = 3$

<u>Ex 3</u>: Find each logarithm. (Remember, a logarithm is an exponent! <u>You are asked to find an exponent, if it exists.</u>) Think: The base to 'what power' equals the argument?

$a)\ \log_2 16 =$ $b)\ \log_5 \left(\frac{1}{5}\right) =$ $c)\ \log_{10} 10000 =$

$d)\ \log_3 (-81) =$ $e)\ \log_4 64 =$ $f)\ \log_{(1/2)} 8 =$

$g)\ \log_7 1 =$ $h)\ \log_2 32 =$ $i)\ \log_{10} 100000 =$

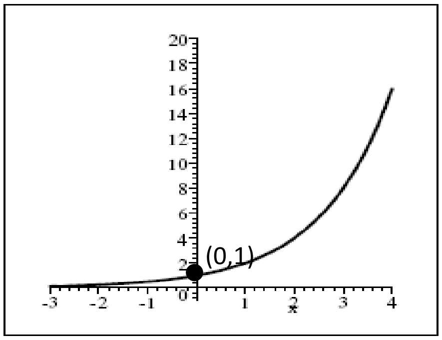
$j)\ \log_{\frac{1}{3}} \frac{1}{27} =$ $k)\ \log_5 625 =$ $l)\ \log_4 \frac{1}{4} =$

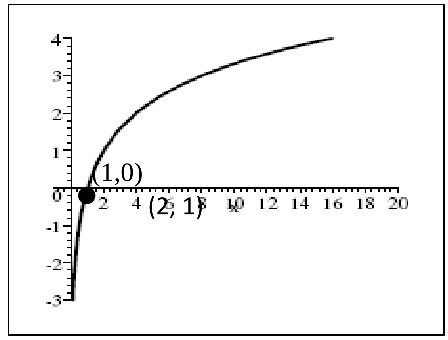
$m)\ \log_9 729 =$ $n)\ \log_{0.1} 0.001 =$ $o)\ \log_{10} \frac{1}{10} =$

$p)\ \log_3 81 =$ $q)\ \log_{20} 0 =$ $r)\ \log_{15} 15 =$

$s)\ \log_{12} \frac{1}{144} =$ $t)\ \log_4 64 =$ $u)\ \log_{\frac{1}{2}} 16 =$

Below is a graph of $y = 2^x$ and its inverse, $x = 2^y$ *or* $y = \log_2 x$.

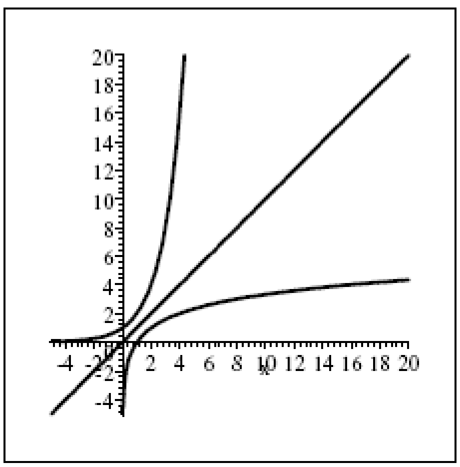




$y = 2^x$ $\qquad$ $x = 2^y$ or $y = log_2 x$

If you imagine the line *y* = *x*, you can see the symmetry about that line. A function and its inverse will have symmetry about the line *y* = *x*.

Below are both graphs on the same coordinate system along with the line *y* = *x*.



There is another graph of $f(x) = y = 2^x$ and the inverse, $f^{-1}(x) = \log_2 x$ on page 91 (calculus part of textbook). Again, you can see the symmetry about the line *y* = *x*.

PROPERTIES OF LOGARITHMS: Let x and y be any positive real numbers and r be any real number. Let a be a positive real number other than 1 ($a \neq 1$). Then the following properties exist.

$a.\quad \log_a xy = \log_a x + \log_a y$

$b.\quad \log_a \frac{x}{y} = \log_a x - \log_a y$

$c.\quad \log_a x^r = r \log_a x$

$d.\quad \log_a a = 1$

$e.\quad \log_a 1 = 0$

$f.\quad \log_a a^r = r$

<u>If you remember that logarithms are exponents, it is easy to understand these properties</u>. See the box below!

The properties of logarithms are easy to prove, if you remember that a logarithm is an exponent. **Logarithms 'behave' like exponents**. When multiplying, exponents are added (property *a*). When dividing, exponents are subtracted (property *b*). When a power is raised to another power, the exponents are multiplied (property *c*). Any real number to the first power is itself (property *d*). Any real number to the zero power is 1 (property *e*). To prove property *f*, just put the logarithmic expression in exponential form.

<u>**Ex 4**</u>: Use the properties of logarithms to write the expression as a sum, difference, or product of simpler logarithms. (**Some textbooks refer to this as 'expanding' the logarithm.)**

$a)\quad \log_2(5\sqrt{x})$

$b)\quad \log_b\left(\frac{2x}{yz^2}\right)$

$c)\quad \log_5\left(\frac{1}{625x^3y^2}\right)$

Ex 4.5: Use the properties of logarithms to write each expression as a single logarithm. Some textbooks call this process **condensing the logarithm.**

$a)$ $\log_3(x+2)+\log_3(x-1)=$

$b)$ $\log_2(x+5)-3\log_2 x=$

$c)$ $3\log x+\log(x+1)=$

Ex 5: Suppose $\log_a 3=m$, $\log_a 4=n$, and $\log_a 5=r$. Use the properties of logarithms to find the following.

$a)$ $\log_a 12$

Hint: Use $12=3\cdot 4$

$b)$ $\log_a \dfrac{25}{3}$

Hint: Use $25=5^2$

$c)$ $\log_a 48a^2$

Hint: Use $48=4^2\cdot 3$

Your scientific 1-line calculator (TI-30 XA) will find logarithms using base 10 (**common logarithms**) or base *e* (**natural logarithms**). If a logarithm is written $\log x$ (with no base indicated), it is assumed to be a common logarithm, or base 10 logarithm. If a logarithm is written $\ln x$, it is assumed to be a natural logarithm (base *e* logarithm). Use your calculator to approximate each of the following to 4 decimal places. (Enter the number, then press either the log key or the ln key.)

$a)$ $\ln 22$ $b)$ $\log 49$

$c)$ $\ln 0.052$ $d)$ $\log 3.2$

Change of base Theorem for Logarithms:
If x is any positive number and if a and b are positive real numbers, $a \neq 1$, $b \neq 1$, then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Most often, base b is chosen to be 10 or e, if a calculator will be used to approximate.

$$\log_a x = \frac{\log x}{\log a} \text{ or } \log_a x = \frac{\ln x}{\ln a}$$

<u>The theorem above is the procedure using a calculator to approximate a logarithm of any base other than 10 or *e*.</u>

<u>Ex 6</u>: Use natural logarithms to evaluate the logarithm. Give an exact answer and an approximation to the nearest thousandth. <u>Do not round until the very end.</u>

$a)$ $\log_4 20$

$b)$ $\log_{\left(\frac{1}{3}\right)} 0.5$

$c)$ $\log_{19}(0.03)$

$d)$ $\log_{1.2} 5$

<u>Solving some simple logarithmic equations:</u>

<u>Ex 7:</u> Solve each equation. Hint: Convert to exponential form. Remember you can only find logarithms of positive numbers, so <u>check your answers</u>. Logarithmic equations **must have potential solutions checked or verified.**

$a)\quad \log_x 64 = 6$

$b)\quad \log_x 27 = -3$

$c)\quad \log_8\left(\frac{1}{64}\right) = x$

$d)\quad \log_3(2x-5) = 2$

$e)\quad \log(x+5)+\log(x+2) = 1$

Review: We have already discussed how to solve some exponential equations by writing each side with the same base. For example: $9^{x-2} = 27^{3x-1}$ can be solved by writing each side with base 3.

$$(3^2)^{x-2} = (3^3)^{3x-1}$$
$$3^{2(x-2)} = 3^{3(3x-1)}$$
$$2(x-2) = 3(3x-1)$$
$$2x-4 = 9x-3$$
$$-1 = 7x$$
$$-\frac{1}{7} = x$$

How can we solve such an equation, if the left and right sides cannot be written with the same base? We will take the natural logarithm (or common logarithm) of both sides and use the properties of logarithms.

<u>Solving some simple exponential equations:</u>

<u>Ex 8</u>: Solve each equation by using natural logarithms (take the natural log of both sides). Approximate to four decimal points, if needed.

$a)\quad 6^{x} = 15$ $\qquad\qquad b)\quad e^{k-2} = 4$

$c)\quad 4^{2x+3} = 6^{x-1}$

Applied problems:

<u>Ex 9:</u> Leigh plans to invest $1000 into an account. Find the interest rate that is needed for the money to grow to $1500 in 8 years if the interest is <u>compounded continuously</u>. Round your percent to the nearest thousandth.

Ex 10: The magnitude of an earthquake, measured on the Richter scale, is given by $R(I) = \log\left(\frac{I}{I_0}\right)$ where *I* is the amplitude registered on a seismograph located 100 km from the epicenter of the earthquake and I_0 is the amplitude of a certain small size earthquake. Find the Richter scale rating of an earthquake with the following amplitude. (Round to the nearest tenth.)

a) $25000I_0$ b) $100{,}000I_0$

Ex 11: The intensity of sound is measured in units called decibels. The higher the decibel rating, the louder the sound. A very faint sound, called the *threshold sound* is assigned an intensity of I_0. If a particular sound has intensity *I,* then the decibel rating of this louder sound is $10 log \frac{I}{I_0}$. Find the decibel ratings of the following sounds having intensities as given. Round answers to the nearest whole number.

a. A soft voice: $250I_0$

b. A loud car 30 feet away: $1{,}500{,}000{,}000I_0$

c. A music concert: $96{,}000{,}000{,}000I_0$

d. An airplane taking off: $445{,}000{,}000{,}000I_0$