MA 15910 Lesson 26 Notes
Graphs of Rational Functions (Asymptotes)
Limits at infinity

Definition of a Rational Function:
If $P(x)$ and $Q(x)$ are both polynomial functions, $Q(x) \neq 0$, then the function f below is called a **Rational Function.**

$$f(x) = \frac{P(x)}{Q(x)}$$

The following are examples of rational functions.

$$f(x) = \frac{2}{x}, \quad g(x) = \frac{x-5}{x+3}, \quad h(x) = \frac{(x+3)(x-3)(x+5)}{(x+6)^2}, \quad j(x) = \frac{x^5 + 3x^3}{2x^4 - 4x^3 + 5x - 1}$$

If there are any values of x for which the denominator of the function would equal 0, these values must be excluded from the domain of the function. At those values of x, the function value does not exist or is not defined and the graph of the function is discontinuous at that value (has a break in the graph).

Any values of $Q(x)$ for which $\underline{Q(x) = 0,}$ but $\underline{P(x) \neq 0}$ (after any common factor has been canceled) are values where there is/are a **vertical asymptote(s).** This is a vertical line where the graph of f (function values) approaches $\infty \; or \; -\infty$ as x goes toward the number we will call c (any number that makes the denominator equal 0).. This is because $\lim\limits_{x \to c} f(x) = \infty$ or $-\infty$. Since it is a vertical line, its equation is $x = c$.

Find the equation(s) of any vertical asymptotes for each rational function.

Ex. 1: $f(x) = \dfrac{12}{x-3}$

Ex. 2: $g(x) = \dfrac{x-5}{x^2 + 7x + 10}$

Ex 3: $y = \dfrac{x^2 - 3x - 10}{x^2 + 2x + 1}$

Ex 4: $j(x) = \dfrac{x+2}{x^3 - x^2 - 12x}$

Ex 5: $r(x) = \dfrac{x^2 + 7x - 18}{(x+1)(x+9)}$

Note: In example 5 of the previous page, there was a factor that 'cancelled out' in both the numerator and denominator. That factor would not give a vertical asymptote. **IN FACT: THERE IS A HOLE IN THE GRAPH AT THE VALUE THAT MAKES THAT FACTOR BE ZERO.**

Find the following limits by dividing each term in the numerator and denominator by the highest power of x in the denominator. Also use the fact that the value of $\dfrac{1}{x^n}$ approaches 0 as x goes toward either ∞ or $-\infty$. (As a denominator gets larger and larger, but the numerator stays relative smaller; the value of the fraction is almost zero.)

$$\boxed{\lim_{n \to \infty} \frac{1}{n} = 0 \text{ and } \lim_{n \to -\infty} \frac{1}{n} = 0}$$ **This is the basic theorem for limits at infinity.**

Find the following limits, if they exist. If a limit goes to ∞ or $-\infty$ (gets larger without bound or smaller without bound), you may say the limit equals to ∞ or $-\infty$.

Ex 6:

$$\lim_{x \to \infty} \frac{3x^4 - 2x}{5x^4 - 7x^3 + 5x^2 - x} = \lim_{x \to \infty} \frac{\dfrac{3x^4}{x^4} - \dfrac{2x}{x^4}}{\dfrac{5x^4}{x^4} - \dfrac{7x^3}{x^4} + \dfrac{5x^2}{x^4} - \dfrac{x}{x^4}}$$

$$= \lim_{x \to \infty} \frac{3 - \dfrac{2x}{x^4}}{5 - \dfrac{7x^3}{x^4} + \dfrac{5x^2}{x^4} - \dfrac{x}{x^4}}$$

$$= \frac{3 - 0}{5 - 0 + 0 - 0}$$

$$= \frac{3}{5}$$

Ex 7:

$$\lim_{a \to (-\infty)} \frac{2a - 4a^2 + 9a^5}{2 - 3a^3 - 6a^5}$$

Ex 8:

$$\lim_{x \to \infty} \frac{2x - 3}{x^2 - 5x + 6}$$

Ex 9:

$$\lim_{x \to (-\infty)} \frac{5x^6 + 2x^4 - 3x^2}{5x^2 - 3x + 2}$$

In examples 6 and 7 of the previous page, <u>the leading term of the numerator had the same degree as the leading term of the denominator</u>. The limit was the ratio of the coefficients of those leading terms.
In example 8 on the previous page, <u>the leading term of the numerator had a lesser degree than the leading term in the denominator</u>. The limit was zero.
In example 9 on the previous page, <u>the leading term of the numerator had a greater degree than the leading term in the denominator</u>. The limit was ∞.

Finding a horizontal asymptote of a <u>rational</u> function (horizontal line that the graph approaches as x goes to ∞ or -∞) uses the same procedure as the limits at $\pm\infty$ in examples 6 through 9. Here is an easy way to determine these limits or the equation of any horizontal asymptote.

1.   If the numerator has a lesser degree than the denominator, there is a horizontal asymptote with equation $y = 0$ (the x-axis).
2.   If the numerator and denominator have the same degree, there is a horizontal asymptote with equation $y =$ the ratio of the leading coefficients.
3.   If the numerator has a greater degree than the denominator, there is no horizontal asymptote.

<u>Note</u>: <u>We will **not be discussing **oblique asymptotes** (lines other than vertical or horizontal that a graph would approach</u>; see example 7 on page 781 of text). One of the lines that the graph approaches is a 'slanted' line. **On any assignment problems in MyMathLab that ask for equations of oblique asymptotes, answer NO. You are not responsible to find any oblique asymptotes on homework problems (paper or MyMathLab).**

<u>**Ex 10**</u>: Find the equations of any horizontal or vertical asymptotes for these rational functions.

a) $g(x) = \dfrac{12x^4 - 5x^2 + 7}{9x - 3x^4}$

b) $F(x) = \dfrac{2x^3 - 4x}{5x^2 + 6x + 1}$

c) $h(x) = \dfrac{9x + 3}{5x^2 - 2x}$

Note: Equations for horizontal asymptotes correspond to limits at infinity or negative infinity. Even though we do not usually describe a limit as ∞ (only a number), in MyMathLab, they may except ∞ as the answer for a limit.

Ex 11: Find the following limits. It a limit is 'unbounded', you may write ∞ or -∞ for the limit value.

a) $\displaystyle\lim_{x \to \infty} \frac{2x-3}{7x+5}$

b) $\displaystyle\lim_{n \to (-\infty)} \frac{4n^2 - 2n + 9}{3n+5}$

c) $\displaystyle\lim_{x \to \infty} \frac{2x^2 - 7x + 7}{3x^3 - 5x^2 + x}$

Optional: Sketch a graph of f using asymptotes, intercepts, and perhaps plotting a few points.

$$f(x) = \frac{6x+3}{2x-4}$$