MA 15910 Notes, Lesson 19
Textbook (calculus part) Section 2.4
Exponential Functions

We have already discussed power functions, such as $f(x) = x^3$ or $g(x) = 5x^4$ In a power function the base is the variable and the exponent is a real number. This lesson covers **exponential functions.** In an exponential function, the variable is in the exponent and the base is a positive constant (other than the number 1).

> **Exponential Function:** An exponential function with base a is defined as $f(x) = a^x$, where $a > 0$ and $a \neq 1$. (Variable is in the exponent. Base is a positive number other than 1.)

> The following are **not** exponential functions. Why?
> $f(x) = x^3$ $f(x) = 1^x$
> $f(x) = (-4)^x$ $f(x) = x^x$

*Note: If a (the base) in the above definition was 1, the function would be constant; a horizontal line, $y = 1$.

Exponential functions often describe what is called exponential growth or exponential decay in real life examples.

An **Example** of an exponential function:
Many real life situations model exponential functions. One example models the average amount spent (to the nearest dollar) by a person at a shopping mall after x hours and is the function, $f(x) = 42.2(1.56)^x$, domain of $x > 1$. The base of this function is 1.56. Notice there is also a 'constant' (42.2) multiplied by the power. Be sure to follow the order of operations; find the exponent power first, then multiply that answer by the 42.2.

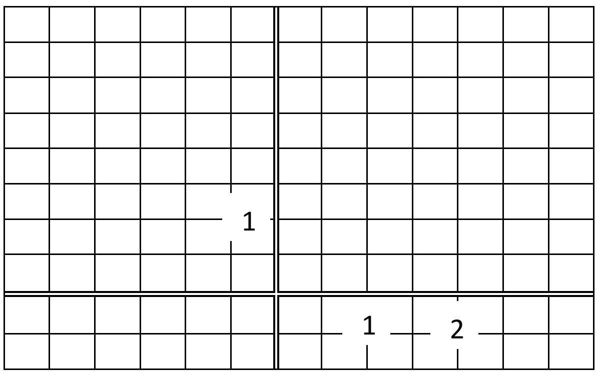
> Suppose you wanted to find the average amount spent in a mall after browsing for 3 hours.
> Let $x = 3$.
> $$\begin{aligned} f(3) &= 42.2(1.56)^3 \\ &= 42.2(3.796416) \\ &= 160.2087552 \end{aligned}$$
> To the nearest dollar, a person in a shopping mall for 3 hours on average would spend $160.

Graphing Exponential Functions:
To graph an exponential function, make a table of ordered pairs as you have for other types of graphs. Notice: If $x = 0$ for b^x , the value is 1 (zero power is 1). For a basic exponential function of the form $y = b^x$, **the *y*-intercept is 1.**

Also, notice **that *y* values will always be positive**, so the graph always lays above the *x*-axis.

Graph each exponential function.

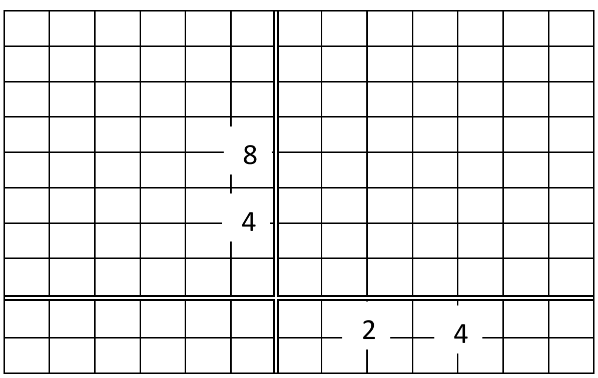
Notice: I changed the scale on each axis below.



$a)\quad y = \left(\frac{3}{2}\right)^x$

x	y
-2	
-1	
0	
1	
2	

Notice: I changed the scale on the y-axis below.



$b)\quad f(x) = \left(\frac{1}{3}\right)^x$

x	y
-1	
-2	
0	
1	
2	

What do you notice about the graphs above?

1)____

2)____

3)____

4)____

Characteristics of Exponential Functions of the form $f(x) = a^x$ (basic exponential function)

1. The **domain** of the function is all real numbers $(-\infty, \infty)$ and the **range** is all positive real numbers $(0, \infty)$ (graph always lies above the *x*-axis).
2. Such a graph will always pass through the **point (0, 1**) and the ***y*-intercept is 1**. There will be **no *x*-intercept.**
3. If the base *a* is **greater than 1** $(a > 1)$**, the graph increases** left to right and is an increasing function. The greater the value of *a*, the steeper the increase (exponential growth).
4. If the base is **between 0 and 1** $(0 < a < 1)$**, the graph decreases** left to right and is a decreasing function (exponential decay). The smaller the value of *a*, the steeper the decrease (exponential decay).
5. The graph represents a 1-1 function and therefore will have an inverse.
6. The graph approaches but does not touch the *x*-axis. The *x*-axis is known as an **asymptote**.

Solving exponential equations: There are a couple of ways to solve equations with the variable in an exponent. The first way is to rewrite both sides of the equation so the bases are the same.

Ex 1: Solve this equation: $25^{\frac{x}{2}} = 125^{x+3}$

Ex 2: Solve: $32^{3x-1} = 16^{5-9x}$

Compound Interest:

One of the best examples in real life where an exponential function is used is in the banking business, compound interest. You know the simple interest formula, $I = Prt$. However, most banks periodically determine interest and add to the account. The amount in an account with an initial amount of *P* dollars invested at an annual interest rate of *r* (as a decimal), compounded *m* (or *n*) times per year for *t* years, yields a compound amount given by either formula below.

$$A = P\left(1+\frac{r}{m}\right)^{mt} \text{ or } A = P\left(1+\frac{r}{n}\right)^{nt}$$

You will need to know how to use the 'power' key on your TI-30XA scientific calculator when you use this formula.

A = the final accumulated amount in the account
P = initial amount (principle)
r = annual interest rate (as a decimal)
t = time in years
m (or *n*) = the number of compounding periods in a year

Ex 2: Find the final amount and the **interest earned** on \$5800 at 4.3% interest compounded semiannually for 6 years.

Definition of the number *e*: Letting P equal \$1 and letting r be 100% and t equal 1 year.

$A = P\left(1+\frac{r}{m}\right)^{mt}$ becomes $\left(1+\frac{1}{m}\right)^{m}$ Let the value of m become extremely large. Then $\left(1+\frac{1}{m}\right)^{m}$ becomes closer and closer to a number we call e, whose approximate value is 2.718281828**. (<u>To find the value of *e* on a TI-30XA calculator use these steps</u>: Enter the number 1, press the 2nd key then the LN key (notice that e^x is above the LN key, so we are finding e^1. You should get the approximation given above.**

As the amount of money in an account is compounded continuously, rather than periodically as with the compound interest formula earlier; the formula uses this number e.

Continuous Compounding Interest: $A = Pe^{rt}$

(The steps to convert the regular compound interest formula to the formula above are shown on pages 82-83 in the textbook. It is a difficult process, so I will not demonstrate it in class. You can examine it on these pages, if you are interested.)

Ex 3: Suppose \$800 is invested at 4 ½ % interest for 5 years. Find the accumulated amount in the account and the interest earned if…

(a) the money is compounded quarterly.

(b) the money is compounded continuously.

Ex 4:

Suppose that a certain type of bacteria grows rapidly in a warm spot. If 500 of these bacteria are placed in a dish in a warm location, and the number present after x hours is given by the model $P(x) = 500 \cdot 3^{3x}$.

Find the number of bacteria after (a) 1 hour, (b) 3 hours.

Ex 5:

Michael plans to invest \$1000 in an investment. What interest rate would be needed for the money to grow to \$1300 in 20 years, if the interest is compounded semiannually? Round to the nearest tenth of a percent.