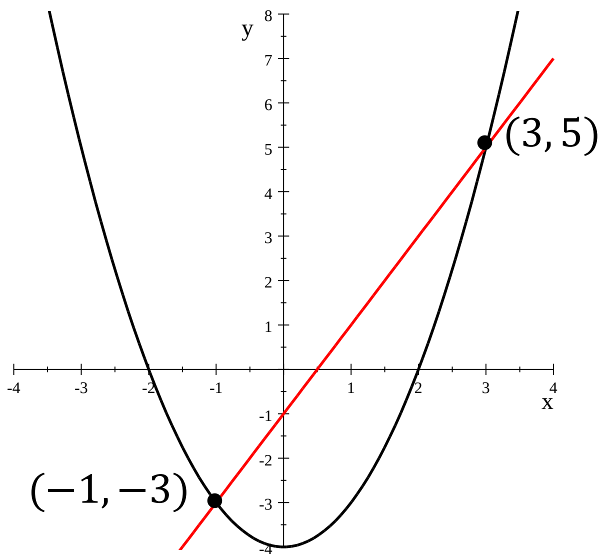
MA 15800, Summer 2016
Lesson 25 Notes
Solving a System of Equations by substitution (or elimination)
Matrices

Consider the graphs of the two equations below.

$$\begin{cases} y = x^2 - 4 \\ y = 2x - 1 \end{cases}$$ A System of Equations

From your mathematics' experience, you probably know that the first equation represents a parabola and the second equation represents a line. If the graphs of these two equations intersect, the ordered pair(s) is(are) called the solution(s) of the **system of equations.**

Below are the graphs of both equations with the two solutions shown. The graph of the first equation, the parabola, is in black and the graph of the second equation, the line, is in red.



The ordered pairs above not only represent the points where the graphs of the two equations intersect, they are also the only ordered pairs that satisfy both equations. In other words, they are the only ordered pairs that make both equations true when the variables *x* and *y* are replaced.

$$\begin{cases} y = x^2 - 4 \\ y = 2x - 1 \end{cases}$$

Using the point (3, 5):

$$5 = 3^2 - 4$$
$$5 = 2(3) - 1$$
✓

Using the point $(-1, -3)$

$$-3 = (-1)^2 - 4$$
$$-3 = 2(-1) - 1$$
✓

A solution to a system of two equations makes both equations true.

Hint: Before solving a system of equations always 'think' about 'how many' solutions there may be. For example: A line and a parabola could intersect at two points (as above) or only one point or at no points (empty set).

Substitution Method:

1. Solve one of the equations for one variable in terms of the other variable.
2. Substitute the expression found in step one into the other equation, obtaining an equation of one variable.
3. Solve for this variable.
4. Back-substitute into the **simplest equation** to find the remaining variable(s).
5. Check each ordered pair in the given system

<u>Ex 1</u>: Solve the system below. Write the solution(s) as ordered pair(s). Verify the solution by graphing each equation.

$$\begin{cases} x+2y=-1 \\ 2x-3y=12 \end{cases}$$

I will solve the first equation for *x* and substitute that expression for *x* in the second equation.

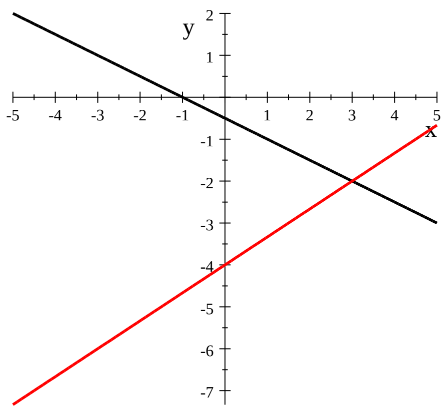
$$\begin{cases} x+2y=-1 \\ 2x-3y=12 \end{cases} \quad \to x=-2y-2$$

$$\to 2(-2y-1)-3y=12$$
$$-4y-2-3y=12$$
$$-7y=14$$
$$y=-2$$

$$\to x+2y=-1$$
$$x+2(-2)=-1$$
$$x-4=-1$$
$$x=3$$

Solution: $(3,-2)$

Computerized graph: Intersection of the two lines is the solution.



First equation's graph is in black; the second equation's graph is in red. Intersection of the two lines is (3, -2).

Note: When using the substitution method and both variables 'drop out' when solving and a false statement, such as 5 = 4, results, there is **no solution. In other words, the graphs of these two equations do not intersect and there are no ordered pair that will satisfy both equations.**

Solve the following systems of equations using the substitution method.

<u>Ex 2:</u> Note: When solving this system of equations, the quadratic equation will be used.

$$\begin{cases} x = y^2 \\ x + 2y + 3 = 0 \end{cases}$$

Substitute the y^2 for the x in the bottom equation.

$y^2 + 2y + 3 = 0$ I will use the quadratic formula to solve.

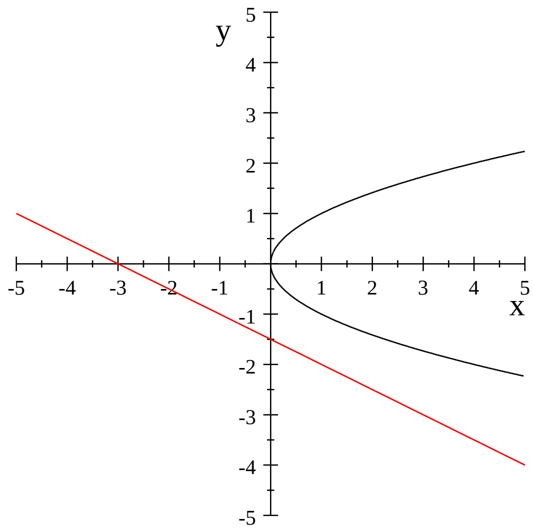
$$y = \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} = \frac{-2 \pm \sqrt{4-12}}{2} = \frac{-2 \pm \sqrt{-8}}{2}$$

There are no real solutions to this equation.

Therefore, this system of equations has NO SOLUTION.

Verify by graphing that the graphs of these two equations do not intersect.

If there is no solution, the graphs of the two equations will not intersect. Below are the graphs of both equations. The top equation is in black; it is a parabola opening right. The bottom equation, a line, is in red.



With a system of two **linear equations,** the elimination method may be used.

1. If necessary, multiply one of the equations by a non-zero constant so that the coefficients of one of the variables are opposites.
2. Add the two equations together and solve for one variable.
3. Back-substitute to find the remaining variable.
4. Check your ordered pair in both equations.

Ex 3: Solve, using the elimination method.

$$\begin{cases} x+3y=-1 \\ 6x-3y=15 \end{cases}$$

The coefficients of the y's are already opposites. I will add the two equations and solve for x.

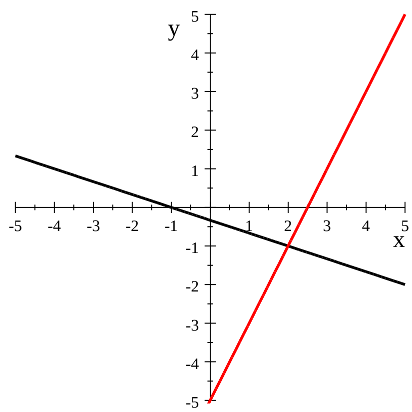
$$\begin{array}{l} x+3y=-1 \\ \underline{6x-3y=15} \\ 7x \qquad =14 \\ \quad x=2 \end{array}$$

Back-substitute in the first equation and solve for y.

$$\begin{aligned} x+3y&=-1 \\ 2+3y&=-1 \\ 3y&=-3 \\ y&=-1 \end{aligned}$$

Solution: $(2,-1)$

The graph below verifies that the solution (intersection of the two lines) is $(2,-1)$. The first equation is in black and the second in red.



We have discussed that there is **no solution** if both variables 'drop out' and a false statement results. When using the elimination method, if both variables 'drop out' and a true statement results (5 = 5, 0 = 0, etc.), then there are an **infinite number of solutions.** This result indicates that the two equations represent the same line. The solution is written in the form $\{(x, y)|\text{ equation of line}\}$. Do not write 'all real numbers'. Solutions are ordered pairs, not numbers.

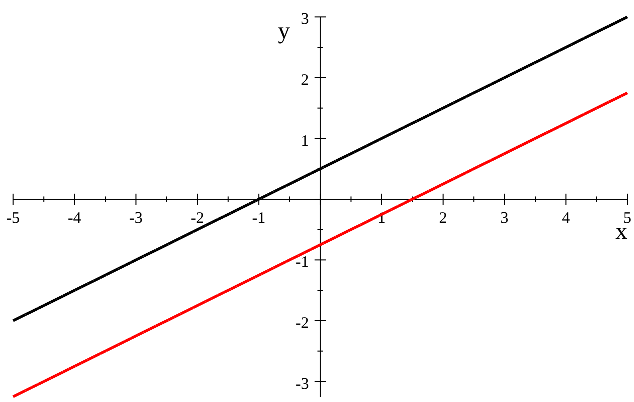
Graphically, this means that the equations represent the same line.

<u>Ex 3.5</u> Solve using the substitution method.

$$\begin{cases} x-2y=-1 \\ 2x-4y=3 \end{cases} \quad \rightarrow x=2y-1 \text{ Substitue for } x \text{ in the other equation.}$$

$$2(2y-1)-4y=3$$
$$4y-2-4y=3$$
$$-2=3$$

As seen above, both variables were eliminated and a false statement resulted. This indicates that there is NO SOLUTION and the graph would be parallel lines. The graph of the lines is seen below. The black line is the first equation; red is the second.



<u>Ex 4:</u> Solve using the elimination method.

$$\begin{cases} 3x+2y=6 \\ 12x+8y=24 \end{cases} \Rightarrow \begin{cases} -4(3x+2y=6) \\ 12x+8y=24 \end{cases} \Rightarrow \begin{cases} -12x-8y=-24 \\ 12x+8y=24 \end{cases}$$

Add the two equations together: $0=0$

As seen above, both variables were eliminated but a TRUE statement resulted. This indicates that there are an INFINITE number of solutions because the two equations are the <u>same line</u>.

$$\{(x,y)\,|\,y=-\frac{3}{2}x+3\} \quad \text{or}$$

The solution would be written $\{(x,y)\,|\,3x+2y=-6\}$ or

$$\{(x,y)\}12x+8y=24$$

Characteristics of a System of 2 Linear Equations in 2 Variable

Graphs	Number of Solutions	Classification
Nonparallel Lines	One	Consistent and Independent System
Same Lines	Infinite	Consistent and Dependent System
Parallel Lines	No Solution	Inconsistent System

Ex 5: Classify the following systems of linear equations as consistent and independent, consistent and dependent, or inconsistent.

a) $\begin{cases} 4x+7y=11 \\ 3x-2y=-9 \end{cases}$

b) $\begin{cases} \frac{1}{3}c+\frac{1}{2}d=5 \\ c-\frac{2}{3}d=-1 \end{cases}$

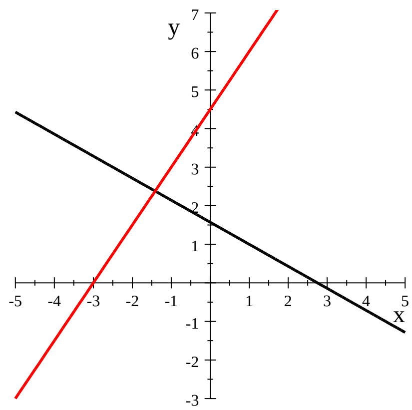
a)

$$\begin{cases} 2(4x+7y=11) \\ 7(3x-2y=-9) \end{cases} \Rightarrow \begin{cases} 8x+14y=22 \\ \underline{21x-14y=-63} \end{cases}$$

$$29x=-14$$

$$x=-\frac{14}{29}$$

There is a solution. Therefore the system is consistent and independent.
Graph of system: First equation is in black, second in red.



b)
First I will 'clear' the denominators in the equations.

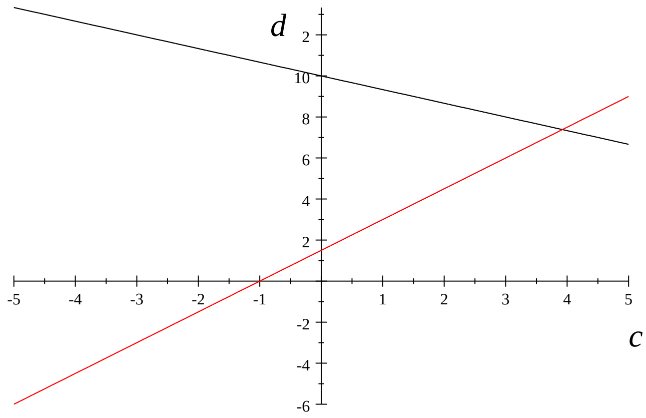
$$\begin{cases} 6(\frac{1}{3}c+\frac{1}{2}d=5) \\ 3(c-\frac{2}{3}d=-1) \end{cases} \Rightarrow \begin{cases} 2c+3d=30 \\ 3c-2d=-3 \end{cases} \Rightarrow \begin{cases} 2(2c+3d=30) \\ 3(3c-2d=-3) \end{cases}$$

$$\Rightarrow \begin{cases} 4c+6d=60 \\ \underline{9c-6d=-9} \end{cases}$$

$$13c=51$$

$$c=\frac{51}{13}$$

There is a solution (variables were not eliminated). Therefore the system is consistent and independent and the graph would be two nonparallel lines.
Below is a graph of both lines.



c) $\begin{cases} 2x - y = 4 \\ 4x - 2y = 1 \end{cases} \Rightarrow \begin{cases} -2(2x - y = 4) \\ 4x - 2y = 1 \end{cases} \Rightarrow \begin{cases} -4x + 2y = -8 \\ \underline{4x - 2y = 1} \end{cases}$

$$0 = -7$$

Variables were eliminated and a false statement resulted. This system has no solution. The graph would be two parallel lines.

Solve each system of equations. If it is a linear system, describe its classification.

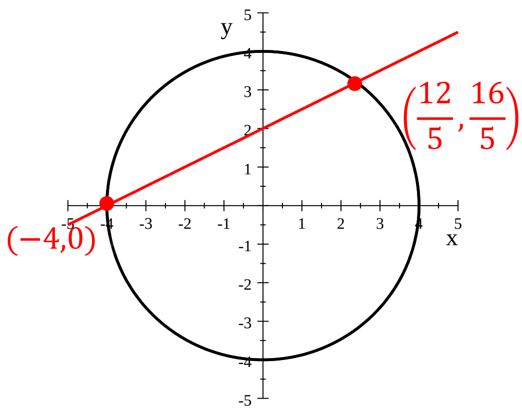
Ex 6: $\begin{cases} x^2 + y^2 = 16 \\ 2y - x = 4 \end{cases}$

I will solve the second equation for *x* and substitute that expression for the *x* in the first equation.

$$2y - 4 = x$$
$$(2y - 4)^2 + y^2 = 16$$
$$4y^2 - 16y + 16 + y^2 = 16$$
$$5y^2 - 16y = 0$$
$$y(5y - 16) = 0$$
$$y = 0 \quad \text{or} \quad 5y - 16 = 0$$
$$y = \frac{16}{5}$$
$$x = 2(0) - 4 \qquad x = 2\left(\frac{16}{5}\right) - 4$$
$$x = -4 \qquad x = \frac{12}{5}$$

Solutions: $(-4, 0), \left(\frac{12}{5}, \frac{16}{5}\right)$

The first equation is a circle and the second is a line. The graphs are shown on the next page.



Ex 7: $\begin{cases} 2y = x^2 \\ y = 4x^3 \end{cases}$

I will substitute the expression for y from the bottom equation for the y in the top equation.

$$2\left(4x^3\right) = x^2$$

$$8x^3 - x^2 = 0$$

$$x^2(8x-1) = 0$$

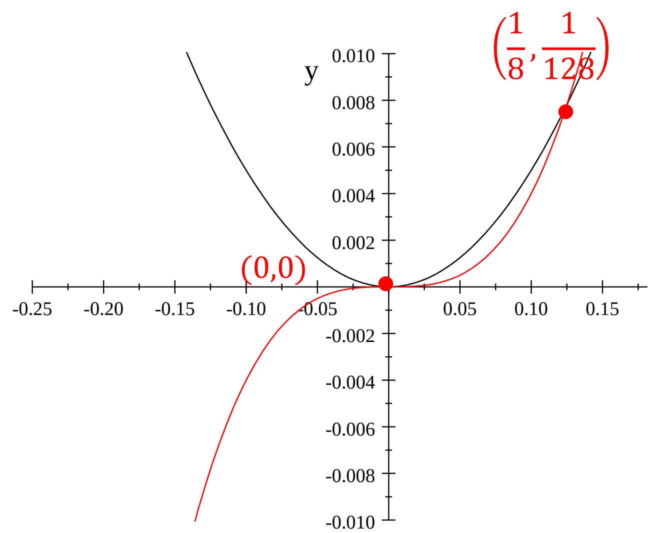
$$x^2 = 0 \quad \text{or} \quad 8x - 1 = 0$$

$$x = 0 \qquad\qquad x = \tfrac{1}{8}$$

The top equation is a parabola. The bottom equation is a cubic curve. Both are graphed below.

$$y = 4(0^3) = 0 \qquad y = 4\left(\tfrac{1}{8}\right)^3 = \frac{1}{128}$$

Solutions: $(0,0)$, $\left(\tfrac{1}{8},\tfrac{1}{128}\right)$



We have solved applied problems where only one equation of one variable was used. Sometimes writing two equations in two variables can be used to solve applied problems.

Ex 8:

The price of admission to a high school play was $3 for students and $4.50 for nonstudents. If 450 tickets were sold for a total of $1555.50, how many of each kind of tickets were purchased?

Let x = the number of student tickets sold

Let y = the number of nonstudent tickets sold

One equation: $x + y = 450$

Another equation: $3x + 4.5y = 1555.5$ I will use the elimination method, multiplying the first equation by -3 then adding so the x terms are eliminated.

$$3(x + y = 450) \qquad -3x - 3y = -1350$$

$$3x + 4.5y = 1555.5 \qquad \underline{3x + 4.5y = 1555.5}$$

$$1.5y = 205.5$$

$$y = 137$$

$$x + y = 450$$

$$x + 137 = 450$$

$$x = 313$$

313 student tickets and 137 nonstudent tickets

Ex 9:

A rancher is preparing an oat-cornmeal mixture for livestock. Each ounce of oats provided 4 grams of protein and 18 grams of carbohydrates. An ounce of cornmeal provides 3 grams of protein and 24 grams of carbohydrates. How many ounces of each can be used to meet the nutritional goals of 200 grams of protein and 1320 grams of carbohydrates per feeding?

Let a = number of ounces of oats

Let b = number of ounces of cornmeal

One equation: $4a + 3b = 200$

Second equation: $18a + 24b = 1320$

I will use the elimination method, multiplying the first equation by -8 and leaving the second equation as is.

$$-8(4a + 3b = 200) \qquad -32a - 24b = -1600$$

$$18a + 24b = 1320 \qquad \underline{18a + 24b = 1320}$$

$$-14a = -280$$

$$a = 20$$

$$4a + 3b = 200$$

$$4(20) + 3b = 200$$

$$80 + 3b = 200$$

$$3b = 120$$

$$b = 40$$

20 ounces of oats and 40 ounces of cornmeal

Ex 10: Solve the following system.

$$\begin{cases}(x-1)^2+(y+2)^2=10\\ x+y=1\end{cases}$$

$y=-x+1$ Substitute in top equation.

$$(x-1)^2+(-x+1+2)^2=10$$
$$(x-1)^2+(3-x)^2=10$$
$$x^2-2x+1+9-6x+x^2=10$$
$$2x^2-8x+10=10$$
$$2x(x-4)=0$$
$$2x=0 \quad \text{or} \quad x-4=0$$
$$x=0 \qquad x=4$$

$$x+y=1$$
$$0+y=1 \qquad 4+y=1$$
$$y=1 \qquad y=-3$$

Solutions: (0,1), (4,–3) Both solutions check in both equations.

The top equation is a circle with the center at $(1,-2)$ and a radius length $\sqrt{10}$. The bottom equation is a line. Both are graphed below and the solutions may be seen.