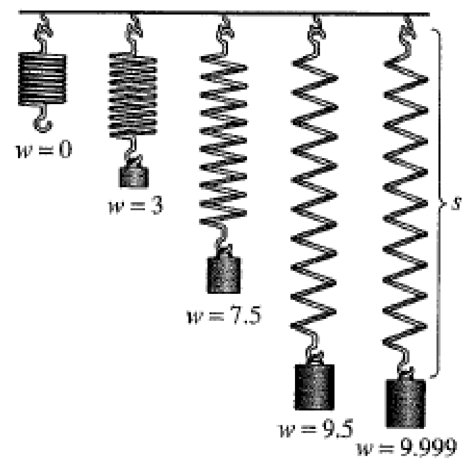
In everyday language, people refer to a speed limit, a wrestler's weight limit, the limit on one's endurance, or stretching a spring to its limit. These phrases all suggest that a limit is a **bound**, which on some occasions may not be reached but on other occasions may be reached or exceeded.

Consider a spring that will break or completely stretch out only if a weight of 10 pounds or more is attached. To determine how far the spring will stretch without breaking, you could attach increasingly heavier weights and measure the 'spring length' s for each weight w, as shown in the picture below. If the spring length approaches a value of L, then it is said that 'the limit of s as w approaches 10 is L.'
A mathematical limit is much like the limit of the spring described. As a weight hanging from a spring approaches 10 pounds, the length of the stretched spring will approach a certain number called the '**limit**'. Let us suppose that limit is 8 inches. (If any more weight than 10 pounds is put on the spring, it will break or stop stretching beyond 8 inches.) We could say 'the limit of the length of the stretched spring as the weight approaches 10 pounds is 8 inches.' This would be written as

$$\lim_{w \to 10} (\text{spring length}) = 8$$

(the limit of the length of the spring as the weight w approaches 10 pounds is 8 inches).



The general limit notation is $\lim_{x \to c} f(x) = L$, which is read 'the limit of $f(x)$ as x approaches c is L.'

Finding Limits:

There are many different strategies used to find limits. One approach is to evaluate the function for numbers very close to *c*, slightly greater than *c* (called approaching from the 'right') and/or slightly less than *c* (called approaching from the 'left'). Sometimes a table is used, such as in the example below.

Ex 1: Find $\lim_{x \to 4} f(x)$ where $f(x) = 2x + 3$ or $\lim_{x \to 4}(2x+3)$.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	10.8	10.98	10.998	11.002	11.02	11.2

Approaching from the left $\rightarrow$ $\leftarrow$ Approaching from the right

You can easily see from the table that the closer the *x* value is to 4, the function value $(2x + 3)$ is approaching 11.

We say $\lim_{x \to 4}(2x+3) = 11$.

As seen above, the first strategy for finding a limit is <u>using a table</u> such as above and approaching the value from the 'left' and from the 'right'.

Coincidently, a 'direct substitution' of 4 into the function expression $2x + 3$ also yields a value of 11.

$$\lim_{x \to 4}(2x+3) = 2(4)+3 = 11$$

A second strategy for finding a limit (works with many polynomial, rational, radical expressions) is <u>**direct substitution**</u>.

Note: When using direct substitution, the limit value is the function value or the *y*-value.

Ex 2: Find $\lim_{x \to 3} g(x)$ where $g(x) = \frac{x^2-9}{x-3}$ or $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

x	2.9	2.99	2.999	3.001	3.01	3.
$g(x)$	5.9	5.99			6.01	6.1

Approaching from the left $\rightarrow$ $\leftarrow$ Approaching from the right

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \boxed{}$$

Notice that direct substitution would not work in example 2 above. Direct substitution yields $\frac{0}{0}$. The expression $\frac{0}{0}$ is called an <u>indeterminant form</u>. Whenever an indeterminant form results, the limit may or may not exist. Another strategy must be tried.

One-sided Limits:

In general for a limit to exist, the same value must be approached 'from the left' and 'from the right'. However, at times, mathematicians and statisticians are only interested in approaching from the left or approaching from the right. These are called one-sided limits, a left-sided limit or a right-sided limit. Below is the notation for these types of limits.

$\lim_{x \to c^-} f(x)$ left-sided limit notation

$\lim_{x \to c^+} f(x)$ right-sided limit notation

Ex 3: Use the table below to determine (if possible) $\lim_{x \to 2^-} \frac{|x-2|}{x-2}$, $\lim_{x \to 2^+} \frac{|x-2|}{x-2}$, and $\lim_{x \to 2} \frac{|x-2|}{x-2}$.

x	1.9	1.99	1.999	2.001	2.01	2.1
$\frac{\lvert x-2\rvert}{x-2}$	-1					1

$\to\to\to\to$(approaching from the left) (approaching from the right) $\leftarrow\leftarrow\leftarrow\leftarrow$

$\lim_{x \to 2^-} \frac{|x-2|}{x-2} =$ $\lim_{x \to 2^+} \frac{|x-2|}{x-2} =$ $\lim_{x \to 2} \frac{|x-2|}{x-2} =$

Ex 4: Use the table below to determine (if possible) $\lim_{x \to 4} \frac{2x-3}{x+1}$.

x	3.9	3.99	3.999	4.001	4.01	4.1
$\frac{2x-3}{x+1}$						

$$\lim_{x \to 4} \frac{2x-3}{x+1} =$$

Ex 5: Use the table below to determine (if possible) $\lim_{x \to -2} \frac{3x+2}{2x+4}$.

x	-2.1	-2.01	-2.001	-2	-1.999	-1.99	-1.9
$\frac{3x+2}{2x+4}$				undefined			

$$\lim_{x \to -2} \frac{3x+2}{2x+4} =$$

Important Conclusions:

1) Saying that the limit of $f(x)$ or an expression as x approaches c is L means that the function or expression value gets very, very close to L as x gets very, very close to c.

2) For a limit L to exist, you must allow x to approach c from either the left side of c or the right side of c and these two values must be the same.

3) The function or expression does not have to be defined at c in order fo the limit as $x \to c$ to exist. In other words, a limit may exist even though the function value does not exist.

Techniques for Finding Limits:

A With many functions or expressions, **a table** similar to those used in previous examples may be used. Approach from the left and from the right. If both are the same value, that is the limit. A table may also be used to find 'one-sided' limits; limits only approaching from the left or only approaching from the right.

B With many functions or expressions, **direct substitution** may be used. Below are a few examples.

$$\lim_{x\to -2}(x^2 - x) = (-2)^2 - (-2) = 4 + 2 = 6$$

$$\lim_{n\to 3}(2n-4)^2 = (2\cdot 3 - 4)^2 = 2^2 = 4$$

$$\lim_{a\to 5}\sqrt{2a+6} = \sqrt{2\cdot 5+6} = \sqrt{16} = 4$$

$$\lim_{n\to 2}\frac{2n-5}{n+1} = \frac{2(2)-5}{2+1} = \frac{-1}{3} \text{ or } -\frac{1}{3}$$

C If direct substitution yields the inderterminant form $\frac{0}{0}$, you can sometimes write an equivalent expression by simplifying and then try direct substitution. Below are a few examples.

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{0}{0} \quad \text{Try simplifying the rational expression first.}$$

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \lim_{x\to 3}\frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

$$\lim_{c\to 5}\frac{c^2-4c-5}{c-5} = \frac{5^2-4(5)-5}{5-5} = \frac{0}{0} \quad \text{Try simplifying the rational expression first.}$$

$$\lim_{c\to 5}\frac{c^2-4c-5}{c-5} = \lim_{c\to 5}\frac{(c-5)(c+1)}{c-5} = \lim_{c\to 5}(c+1) = 5+1 = 6$$

$$\lim_{\Delta x\to 0}\left(\frac{3(x+\Delta x)-2-(3x-2)}{\Delta x}\right) = \frac{3(x+0)-2-(3x-2)}{0} = \frac{0}{0} \quad \text{Try simplifying first.}$$

$$\lim_{\Delta x\to 0}\left(\frac{3(x+\Delta x)-2-(3x-2)}{\Delta x}\right) = \lim_{\Delta x\to 0}\frac{3x+3\Delta x-2-3x+2}{\Delta x} = \lim_{\Delta x\to 0}\frac{3\Delta x}{\Delta x} = \lim_{\Delta x\to 0}3 = 3$$

Ex 6: Find each limit, if possible. If the limit does not exist, write DNE.

a) $\lim_{x \to 5} (2x+3) =$

b) $\lim_{a \to -3} (2a^2 - 5a + 7) =$

c) $\lim_{x \to -30} \sqrt{21+x} =$

d) $\lim_{m \to 2} \frac{\frac{1}{m} - \frac{1}{m+2}}{m} =$

e) $\lim_{x \to -10} \frac{x^2 - 100}{x+10} =$

f) $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6} =$

g) $\lim_{x \to 0} \frac{\frac{1}{2} - \frac{1}{x+2}}{x} =$

h) $\lim_{x \to 4} \frac{x-4}{x^2 - 8x + 16} =$

i) $\lim\limits_{t\to 2}\dfrac{t^2+3t-10}{t^2-4}=$

j) $\lim\limits_{c\to 0}\left(\dfrac{4(x+c)+3-(4x+3)}{c}\right)=$

k) $\lim\limits_{\Delta r\to 0}\left(\dfrac{(r+\Delta r)^2-2(r+\Delta r)-1-(r^2-2r-1)}{\Delta r}\right)=$

Ex 7: Find the following one-sided limits.

a) $\lim\limits_{x\to(-1)^-}\dfrac{x^2+x-6}{x^2-9}=$

b) $\lim\limits_{x\to 0^-}(2\sin x-3)=$

c) $\lim_{c \to 0^+} \frac{7c}{c^2 + 5c} =$

d) $\lim_{x \to 4^-} \frac{-2x}{(x-4)^2} =$

e) $\lim_{x \to 0} \frac{5x^2}{x} =$