Fall 2017

MA 16100
Study Guide - Exam # 2

1 **Basic Differentiation Rules**: If f and g are differentiable functions, and c is a constant:

(a) $\dfrac{d(c)}{dx} = 0$ (b) $\dfrac{d(u+v)}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ (c) $\dfrac{d(u-v)}{dx} = \dfrac{du}{dx} - \dfrac{dv}{dx}$

(c) `Power Rule`: $\dfrac{d(x^n)}{dx} = nx^{n-1}$

(d) `Product Rule` : $\dfrac{d(uv)}{dx} = u\,\dfrac{dv}{dx} + v\,\dfrac{du}{dx}$ `Quotient Rule` : $\dfrac{d\left(\frac{u}{v}\right)}{dx} = \dfrac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$

2 *Special Trig Limits* : $\boxed{\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1}$ $\boxed{\lim_{\theta \to 0} \dfrac{\theta}{\sin\theta} = 1}$ $\boxed{\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0}$.

Hence also $\lim_{\theta \to 0} \dfrac{\sin(k\theta)}{(k\theta)} = 1$ and $\lim_{\theta \to 0} \dfrac{(k\theta)}{\sin(k\theta)} = 1$. Note that $\sin k\theta \neq k \sin\theta$.

3 **CHAIN RULE**: If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $f \circ g$ is differentiable at x and its derivative is

$$(f \circ g)'(x) = \Big\{ f(g(x)) \Big\}' = f'(g(x))\, g'(x)$$

i.e., if $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\,\dfrac{du}{dx}$.

4 **Implicit Differentiation**: If an equation defines one variable as a function of the other (independent) variable, then to find the derivative of the function w.r.t. the independent variable:

`Step 1 -` Differentiate both sides of equation w.r.t. independent variable
`Step 2 -` Solve for the desired derivative

5 **Logarithmic Differentiation**

`Step 1` : Take natural log of both sides of $y = h(x)$; simplify using Law of Logarithms
`Step 2` : Differentiate implicitly w.r.t x
`Step 3` : Solve the resulting equation for $\dfrac{dy}{dx}$

6 Inverse Trig Functions - Note, for example, $\sin^{-1} x$ is same as $\arcsin x$, but $\sin^{-1} x \neq (\sin x)^{-1}$

(a) Definitions:

$y = \sin^{-1} x \iff \sin y = x \quad (-\frac{\pi}{2} \leq x \leq \frac{\pi}{2})$

$y = \cos^{-1} x \iff \cos y = x \quad (0 \leq x \leq \pi)$

$y = \tan^{-1} x \iff \tan y = x \quad (-\frac{\pi}{2} < x < \frac{\pi}{2})$

(b) Basic Derivatives:

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

If u is a differentiable function of x, then by the Chain Rule, $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{1}{\sqrt{1-u^2}} \dfrac{du}{dx}$, etc.

7 Hyperbolic Trig Functions

(a) Definitions:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \sinh x = \frac{e^x - e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

(b) Basic Derivatives:

$$\frac{d(\cosh x)}{dx} = \sinh x \qquad \frac{d(\sinh x)}{dx} = \cosh x \qquad \frac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$$

If u is a differentiable function of x, then by the Chain Rule, $\dfrac{d(\cosh u)}{dx} = (\sinh u) \dfrac{du}{dx}$, etc.

(c) Basic Identities:

$$\cosh(-x) = \cosh x \qquad \sinh(-x) = -\sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

8 APPLICATIONS

`Model 1` - **Exponential Growth/Decay**: $\dfrac{dy}{dt} = k\,y$ where $k =$ `relative growth/decay rate`

(If the rate of change of y is proportional to y, then the above differential equation holds.)

- If $k > 0$, this is the law of *Natural Growth* (for example, population growth).
- If $k < 0$, this is the law of *Natural Decay* (for example, radioactive decay).

All solutions to this differential equation have the form $y(t) = y(0)\, e^{kt}$.

(Usually need **two** pieces of information to determine both constants $y(0)$ and k, unless they are given explicitly.)

Half-life = time it takes for radioactive substance to lose half its mass.)

`Model 2` - **Newton's Law of Cooling** : If $T(t)$ = temperature of an object at time t and T_s = temperature of its surrounding environment, then the rate of change of $T(t)$ is proportional to the difference between $T(t)$ and T_s :

$$\frac{dT}{dt} = k\,(T(t) - T_s)$$

The solution to this particular differential equation is always $T(t) = T_s + Ce^{kt}$

(Usually need **two** pieces of information to determine both constants C and k, unless they are given explicitly.)

Additional Differentiation Formulas

(u is a differentiable function of x)

$\dfrac{d\left(u^n\right)}{dx} = nu^{n-1}\,\dfrac{du}{dx}$	$\dfrac{d\left(e^u\right)}{dx} = e^u\,\dfrac{du}{dx}$	$\dfrac{d\left(a^u\right)}{dx} = a^u\left(\ln a\right)\dfrac{du}{dx}$

$\dfrac{d\left(\ln u\right)}{dx} = \dfrac{1}{u}\,\dfrac{du}{dx}$	$\dfrac{d\left(\log_a u\right)}{dx} = \dfrac{1}{u\ln a}\,\dfrac{du}{dx}$

$\dfrac{d\left(\sin u\right)}{dx} = \left(\cos u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\cos u\right)}{dx} = \left(-\sin u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\tan u\right)}{dx} = \left(\sec^2 u\right)\dfrac{du}{dx}$

$\dfrac{d\left(\csc u\right)}{dx} = \left(-\csc u\cot u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\sec u\right)}{dx} = \left(\sec u\tan u\right)\dfrac{du}{dx}$	$\dfrac{d\left(\cot u\right)}{dx} = \left(-\csc^2 u\right)\dfrac{du}{dx}$