

Study Guide # 2

1. Constrained extreme values via **Lagrange Multipliers**: Max/min -ize $f(\mathbf{v})$ subject to constraint $g(\mathbf{v}) = C$, solve the system $\nabla f = \lambda \nabla g$ and $g(\mathbf{v}) = C$.

2. Double integrals; Double Riemann sums: $\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(x_i^*, y_j^*) \Delta A$;

3. Type I region $D : \begin{cases} g_1(x) \leq y \leq g_2(x) \\ a \leq x \leq b \end{cases}$; Type II region $D : \begin{cases} h_1(y) \leq x \leq h_2(y) \\ c \leq y \leq d \end{cases}$;

iterated integrals over Type I and II regions: $\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx$ and

$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

4. Integral inequalities: $mA \leq \iint_D f(x, y) dA \leq MA$, where $A = \text{area of } D$ and $m \leq f(x, y) \leq M$ on D .

5. Polar: $r^2 = x^2 + y^2$, $x = r \cos \theta$, $y = r \sin \theta$, $\tan \theta = \frac{y}{x}$ (make sure θ in correct quadrant).

Change of Variables Formula in Polar Coordinates: if $D : \begin{cases} h_1(\theta) \leq r \leq h_2(\theta) \\ \alpha \leq \theta \leq \beta \end{cases}$, then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) \underset{\uparrow}{r} dr d\theta.$$

6. Applications of double integrals:

(a) Area of region D is $A(D) = \iint_D dA$

(b) Volume of solid under graph of $z = f(x, y)$, where $f(x, y) \geq 0$, is $V = \iint_D f(x, y) dA$

(c) Mass of D is $m = \iint_D \rho(x, y) dA$, where $\rho(x, y) = \text{density (per unit area)}$; sometimes write $m = \iint_D dm$, where $dm = \rho(x, y) dA$.

(d) Moment about the x -axis $M_x = \iint_D y \rho(x, y) dA$; moment about the y -axis $M_y = \iint_D x \rho(x, y) dA$.

(e) Center of mass $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{M_y}{m} = \frac{\iint_D x \rho(x, y) dA}{\iint_D \rho(x, y) dA}$, $\bar{y} = \frac{M_x}{m} = \frac{\iint_D y \rho(x, y) dA}{\iint_D \rho(x, y) dA}$

(f) Surface Area $A(S) = \iint_D \sqrt{1 + f_x^2 + f_y^2} dA$

7. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E :

$$\iiint_E f(x, y, z) dV = \iint_D \int_{u(x, y)}^{v(x, y)} f(x, y, z) dz dA \text{ for } E = \{(x, y) \in D, u(x, y) \leq z \leq v(x, y)\};$$

volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz} , center of mass of a solid $(\bar{x}, \bar{y}, \bar{z})$.

8. Cylindrical Coordinates (r, θ, z) :

$$\text{From CC to RC : } \begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan \theta = \frac{y}{x}$ (make sure θ is in correct quadrant).

9. Spherical Coordinates (ρ, θ, ϕ) , where $0 \leq \phi \leq \pi$:

$$\text{From SC to RC : } \begin{cases} x = (\rho \sin \phi) \cos \theta \\ y = (\rho \sin \phi) \sin \theta \\ z = \rho \cos \phi \end{cases}$$

Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan \theta = \frac{y}{x}$ and $\cos \phi = \frac{z}{\rho}$.

10. Triple integrals in Cylindrical Coordinates: $\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}, \quad dV = r dz dr d\theta$

$$\iiint_E f(x, y, z) dV = \iiint_E f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

↑

11. Triple integrals in Spherical Coordinates: $\begin{cases} x = (\rho \sin \phi) \cos \theta \\ y = (\rho \sin \phi) \sin \theta \\ z = \rho \cos \phi \end{cases}, \quad dV = \rho^2 \sin \phi d\rho d\phi d\theta$

$$\iiint_E f(x, y, z) dV = \iiint_E f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

↑

12. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ and

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle;$$

$\mathbf{F}$ is a conservative vector field if $\mathbf{F} = \nabla f$, for some real-valued function f (potential).

13. Line integral of a function $f(x, y)$ along C , parameterized by $x = x(t)$, $y = y(t)$ and $a \leq t \leq b$, is

$$\int_C f(x, y) \, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

(independent of orientation of C , other properties and applications of line integrals of f)

Remarks:

(a) $\int_C f(x, y) \, ds$ is sometimes called the “line integral of f with respect to arc length”

(b) $\int_C f(x, y) \, dx = \int_a^b f(x(t), y(t)) x'(t) \, dt$

(c) $\int_C f(x, y) \, dy = \int_a^b f(x(t), y(t)) y'(t) \, dt$

14. Line integral of vector field $\mathbf{F}(x, y)$ along C , parameterized by $\mathbf{r}(t)$ and $a \leq t \leq b$, is given by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.$$

(depends on orientation of C , other properties and applications of line integrals of $\mathbf{F}$)

15. Connection between line integral of vector fields and line integral of functions:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\mathbf{F} \cdot \mathbf{T}) \, ds$$

where $\mathbf{T}$ is the unit tangent vector to the curve C .

16. If $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$, then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P(x, y) \, dx + Q(x, y) \, dy$; Work = $\int_C \mathbf{F} \cdot d\mathbf{r}$.