7. Let $\boldsymbol{w1} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$ and $\boldsymbol{W} = \mathbf{span}\{\boldsymbol{w1}, \boldsymbol{w2}\}$.

a) Show that $\{\boldsymbol{w1}, \boldsymbol{w2}\}$ is an orthonormal set.

b) Use 10.4 to determine $\mathbf{proj}_{\boldsymbol{W}}\boldsymbol{u}$ where $\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

Section 10.3

The Gram – Schmidt Process

The ideas about projections in Section 10.2 actually tell us a way to construct an orthonormal basis from an existing basis provided we build the new basis one vector at a time.

The Gram-Schmidt process takes a basis $\mathbf{S} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_n}\}$ for a subspace of an inner product space $\mathbf{V}$ and produces a new basis $\mathbf{T} = \{\boldsymbol{w_1}, \boldsymbol{w_2}, ..., \boldsymbol{w_n}\}$ whose vectors form an orthonormal set. The process is often performed in two stages:

- First from the **S**-basis generate a basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, ..., \boldsymbol{v_n}\}$ of vectors that are mutually orthogonal. That is, $(\boldsymbol{v_i}, \boldsymbol{v_j}) = 0$, $i \neq j$.
- Second normalize each of the orthogonal basis vectors into a unit vector.

The first stage involves solving a set of equations and the second is easily performed using $\boldsymbol{w_i} = \boldsymbol{v_i} / \parallel \boldsymbol{v_i} \parallel$. At each step in the first stage we use projections onto subspaces.

The First Stage

Step 1. Define $\boldsymbol{v_1} = \boldsymbol{u_1}$.

Step 2. Look for a vector $\boldsymbol{v_2}$ in the $\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{u_2}\}$ that is orthogonal to $\boldsymbol{v_1}$. This will then guarantee that

$$\begin{aligned} \mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}\} &= \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{u_2}\} && \text{since } \boldsymbol{v_1} = \boldsymbol{u_1} \\ &= \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\} && \text{since } \boldsymbol{v_2} \text{ is a linear combination of } \boldsymbol{v_1} \text{and } \boldsymbol{u_2} \end{aligned}$$

Let $\boldsymbol{v_2} = k_1\boldsymbol{v_1} + k_2\boldsymbol{u_2}$. Find k_1 and k_2 so that $(\boldsymbol{v1}, \boldsymbol{v2}) = \mathbf{0}$.

$$0 = (\boldsymbol{v_1}, \boldsymbol{v_2}) = k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) + k_2(\boldsymbol{v_1}, \boldsymbol{u_2})$$

We have one equation in two unknowns, so let $k_2 = 1$ and solve for k_1. We get

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}$$

thus we have

$$\boldsymbol{v_2} = \boldsymbol{u_2} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} = \boldsymbol{u_2} - \mathbf{proj}_{\boldsymbol{v_1}}\, \boldsymbol{u_2}$$

Step 3. Look for a vector $\boldsymbol{v_3}$ in $\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\}$ that is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. This will guarantee that $\mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$. Let $\boldsymbol{v_3} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + k_3\boldsymbol{u_3}$. Find k_1, k_2, and k_3 so that $(\boldsymbol{v_1}, \boldsymbol{v_3}) = \mathbf{0}$ and $(\boldsymbol{v_2}, \boldsymbol{v_3}) = 0$.

$$\begin{aligned} 0 &= (\boldsymbol{v_1}, \boldsymbol{v_3}) = \boldsymbol{k_1}(\boldsymbol{v_1}, \boldsymbol{v_1}) + \boldsymbol{k_2}(\boldsymbol{v_1}, \boldsymbol{v_2}) + \boldsymbol{k_3}(\boldsymbol{v_1}, \boldsymbol{u_3}) \\ 0 &= (\boldsymbol{v_2}, \boldsymbol{v_3}) = \boldsymbol{k_1}(\boldsymbol{v_2}, \boldsymbol{v_1}) + \boldsymbol{k_2}(\boldsymbol{v_2}, \boldsymbol{v_2}) + \boldsymbol{k_3}(\boldsymbol{v_2}, \boldsymbol{u_3}) \end{aligned}$$

Since by construction $(\boldsymbol{v1}, \boldsymbol{v2}) = 0$ the preceding equations simplify to

$$\begin{array}{lll} k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) & & + k_3(\boldsymbol{v_1}, \boldsymbol{u_3}) = 0 \\ & k_2(\boldsymbol{v_2}, \boldsymbol{v_2}) & + k_3(\boldsymbol{v_2}, \boldsymbol{u_3}) = 0 \end{array}$$

Thus we have 2 equations in 3 unknowns. Let $k_3 = 1$, then we find that

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})} \quad \text{and} \quad k_2 = \frac{-(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}$$

and hence

$$\boldsymbol{v_3} = \boldsymbol{u_3} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} - \frac{(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}\boldsymbol{v_2} = \boldsymbol{u_3} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}}\, \boldsymbol{u_3}$$

Other steps: $\boldsymbol{v_k} = \boldsymbol{u_k} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_{k-1}}\}}\, \boldsymbol{u_k}$

The Second Stage

The orthonormal basis for $\mathbf{V}$ is given by

$$\{\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_n}\} = \left\{ \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|}, \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|}, \ldots, \frac{\boldsymbol{v_n}}{\|\boldsymbol{v_n}\|} \right\}$$

Example 1. Let $\mathbf{V}=$ $\mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ where

$$\boldsymbol{u_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \boldsymbol{u_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \boldsymbol{u_3} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Use the Gram-Schmidt process to find an orthonormal basis for $\mathbf{V}$.

Step 1. Define $\boldsymbol{v_1} = \boldsymbol{u_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$.

Step 2. Compute $\boldsymbol{v_2} = \boldsymbol{u_2} - \mathbf{proj}_{\boldsymbol{v_1}}\, \boldsymbol{u_2} = \boldsymbol{u_2} - \frac{(\boldsymbol{v_1},\boldsymbol{u_2})}{(\boldsymbol{v_1},\boldsymbol{v_1})}\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{14}{21}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}$.

Step 3. Compute $\boldsymbol{v_3} = \boldsymbol{u_3} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1},\boldsymbol{v_2}\}}\, \boldsymbol{u_3} = \boldsymbol{u_3} - \frac{(\boldsymbol{v_1},\boldsymbol{u_3})}{(\boldsymbol{v_1},\boldsymbol{v_1})}\boldsymbol{v_1} - \frac{(\boldsymbol{v_2},\boldsymbol{u_3})}{(\boldsymbol{v_2},\boldsymbol{v_2})}\boldsymbol{v_2} =$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{21}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} - \frac{(-2/3)}{15/9}\begin{bmatrix} \frac{-1}{3} \\ \frac{-2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{17}{35} \\ \frac{54}{35} \\ \frac{7}{5} \\ \frac{-22}{35} \end{bmatrix}.$$

The set $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ is an orthogonal basis for $\mathbf{V}$. An orthonormal basis is obtained by dividing each vector by it length.

$$\boldsymbol{w_1} = \frac{\boldsymbol{v_1}}{\sqrt{21}}, \qquad \boldsymbol{w_2} = \frac{\boldsymbol{v_2}}{\sqrt{17/9}}, \qquad \boldsymbol{w_3} = \frac{\boldsymbol{v_3}}{\sqrt{6090/1225}}$$

For $\mathbf{V} = R^n$ and the standard inner product both stages of the Gram-Schmidt process are available in MATLAB routine **gschmidt**. Type **help gschmidt** for more details. The following examples illustrate the use of routine **gschmidt**.

Example 2. Let $\mathbf{S} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis for R^3. To find an orthonormal basis from $\mathbf{S}$ using MATLAB enter the vectors $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ as columns of a matrix $\boldsymbol{A}$ and type

$$\mathbf{B = gschmidt(A)}$$

The display generated is

```
B =

    0.4472    0.7807   -0.4364
         0    0.4880    0.8729
    0.8944   -0.3904    0.2182
```

The columns of $\boldsymbol{B}$ are an orthonormal basis for R^3.

Example 3. We will show how to find an orthonormal basis for R^4 containing scalar multiples of the vectors

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } \boldsymbol{v_2} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

First enter $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ into MATLAB as vectors **v1** and **v2**, respectively. To find a basis containing scalar multiples of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, use commands

A = [v1 v2 eye(4)]
rref(A)

The display indicates that the first four columns of $\boldsymbol{A}$ form a basis for R^4. The command **S = A(:,1:4)** produces the matrix with those columns. Type the command

T = gschmidt(S)

The display is

```
T =
    0.5774   -0.3780    0.7237         0
         0    0.3780    0.1974    0.9045
    0.5774    0.7559   -0.0658   -0.3015
   -0.5774    0.3780    0.6580   -0.3015
```

Column 1 of $\boldsymbol{T}$ is $\left(\frac{1}{\|\boldsymbol{v_1}\|}\right)\boldsymbol{v_1}$ and column 2 of $\boldsymbol{T}$ is $\left(\frac{1}{\|\boldsymbol{v_2}\|}\right)\boldsymbol{v_2}$, hence the columns of $\boldsymbol{T}$ form the desired orthonormal basis for R^4.

Explain what to do if **rref(A)** did not indicate that the first four columns of $\boldsymbol{A}$ form a basis for R^4.

Exercises 10.3

1. Let $\mathbf{V} = R^3$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use routine **gschmidt** in MATLAB to obtain an orthonormal basis $\mathbf{T}$ and then find the coordinates of $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to $\mathbf{T}$. Record the orthonormal basis and the coordinates of $\boldsymbol{x}$ below.

2. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use routine **gschmidt** in MATLAB to obtain an orthonormal basis $\mathbf{T}$ and then find the coordinates of $\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ relative to $\mathbf{T}$. Record the orthonormal basis and the coordinates of $\boldsymbol{x}$ below.

3. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}, \boldsymbol{u_4}\} = \left\{ \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}, \begin{bmatrix} .5 \\ .5 \\ -.5 \\ -.5 \end{bmatrix}, \begin{bmatrix} .5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix}, \begin{bmatrix} .5 \\ -.5 \\ .5 \\ -.5 \end{bmatrix} \right\}.$$

a) Is **S** an orthonormal basis? Circle one: Yes No

Explain your answer.

b) In MATLAB form the matrix $\boldsymbol{T}$ whose columns are the vectors in **S**. Generate a random vector in R^4 using command **x = rand(4,1)** and then compute $\| \boldsymbol{x} \|$ and $\| \boldsymbol{Tx} \|$. How are the values of the norms related? Repeat the experiment for another arbitrary vector.

____

____

4. Let $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. In MATLAB form the matrix **A = [v1 v2]** and then use command **gschmidt(A)**. Explain the meaning of the display generated.

____

____

5. Let $\boldsymbol{A} = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}$.

a) In MATLAB use command **A′**. Record the result. **A′** = ____________

b) In MATLAB use command **C = A′∗A**. Record the result. **C** = ____________

c) What is the relation between $\boldsymbol{C}$ and $\boldsymbol{C'}$?

____

d) Experiment with other complex matrices $\boldsymbol{A}$ to confirm or reject your answer in part c).

Circle one: confirmed not confirmed.

6. A complex matrix $\boldsymbol{A}$ is called Hermitian if it is equal to its conjugate transpose. The command $\mathbf{A}'$ gives the conjugate transpose in MATLAB .

a) How can you use MATLAB to determine if the matrix $\boldsymbol{A}$ below is Hermitian?

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$

____

b) Compute $\boldsymbol{r} = \boldsymbol{x}' * \boldsymbol{A} * \boldsymbol{x}$ for the complex vector below.

$$\boldsymbol{x} = \begin{bmatrix} i \\ 1-i \end{bmatrix} \qquad \boldsymbol{r} = \_\_\_\_\_\_\_\_\_\_\_\_$$

Is $\boldsymbol{r}$ a real number? (Circle one:) YES NO

c) Experiment with other complex vectors $\boldsymbol{x}$ to determine whether $\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}$ will always be a real number. (Circle one:)

Always a real number for this matrix $\boldsymbol{A}$. Not always a real number.

d) Experiment with another Hermitian matrix $\boldsymbol{A}$ and arbitrary vector $\boldsymbol{x}$ to see if $\boldsymbol{r} = \boldsymbol{x}' * \boldsymbol{A} * \boldsymbol{x}$ is always a real number.

(Circle one:) Always a real number. Not always a real number.

7. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$\boldsymbol{v_1} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \quad \boldsymbol{v_2} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v_3} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \end{bmatrix}.$$

a) Find an orthonormal basis for R^4 containing scalar multiples of the vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. Record your result below.

b) Find an orthonormal basis for R^4 containing scalar multiples of the vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$. Record your result below.

<< NOTES; COMMENTS; IDEAS >>