Supplementary Problems

A. For what value(s) of $A$, if any, will $y = Ae^{-2t}$ be a solution of the differential equation $2y' + 4y = 3e^{-2t}$? For what value(s) of $B$, if any, will $y = Be^{-2t}$ be a solution?

B. Using the substitution $u(x) = y + x$, solve the differential equation $\frac{dy}{dx} = (y + x)^2$.

C. Using the substitution $u(x) = y^3$, solve the differential equation $y^2\frac{dy}{dx} + \frac{y^3}{x} = \frac{2}{x^2} (x > 0)$.

D. Find the explicit solution of the Separable Equation $\frac{dy}{dt} = y^2 - 4y$, $y(0) = 8$. What is the largest open interval containing $t = 0$ for which the solution is defined?

E. The graph of $F(y)$ vs $y$ is as shown:

![Graph of F(y) vs y]

(a) Find the equilibrium solutions of the autonomous differential equation $\frac{dy}{dt} = F(y)$.

(b) Determine the stability of each equilibrium solution.

F. Solve the differential equation $\frac{dw}{dt} = \frac{2tw}{w^2 - t^2}$.

G. (a) If $y' = -2y + e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (eul) with step sizes of the form $h = 1/n$ to find the smallest integer $n$ which will give a value $y_n$ that approximates the above true solution at $t = 1$ within $0.05$.

H. (a) If $y' = 2y - 3e^{-t}$, $y(0) = 1$ then compute $y(1)$.

(b) Experiment using the Euler Method (eul) with step sizes of the form $h = 1/n$ to find the smallest integer $n$ which will give a value $y_n$ that approximates the above true solution at $t = 1$ within $0.05$.

I. Approximation methods for differential equations can be used to estimate definite integrals:

(a) Show that $y(t) = \int_0^t e^{-u^2} du$ satisfies the initial value problem $\frac{dy}{dt} = e^{-t^2}$, $y(0) = 0$. 


(b) Use the Euler Method (eu1) with $h = 1/2$ to approximate the integral $\int_0^2 e^{-u^2} du$.

J. Given that the general solution to $t^2y'' - 4ty' + 4y = 0$ is $y = C_1t + C_2t^4$, solve the following initial value problem:

\[
\begin{align*}
    t^2y'' - 4ty' + 4y &= -2t^2 \\
y(1) &= 2, \quad y'(1) = 0
\end{align*}
\]

K. From the theory of elasticity, if the ends of a horizontal beam (of uniform cross-section and constant density) are supported at the same height in vertical walls, then its vertical displacement $y(x)$ satisfies the Boundary Value Problem

\[
\begin{align*}
y^{(4)} &= -P \\
y(0) = y(L) &= 0 \\
y'(0) = y'(L) &= 0
\end{align*}
\]

where $P > 0$ is a constant depending on the beam’s density and rigidity and $L$ is the distance between supporting walls:

![ Beam Diagram ]

(a) Solve the above boundary value problem when $L = 4$ and $P = 24$.

(b) Show that the maximum displacement occurs at the center of the beam $x = \frac{L}{2} = 2$.

L. Laplace transforms may be used to find particular solutions to some nonhomogeneous differential equations. Use Laplace transforms to find a particular solution, $y_p(t)$, of $y'' + 4y = 20e^t$.

**Hint:** Solve the initial value problem

\[
\begin{align*}
y'' + 4y &= 20e^t \\
y(0) = 0, \quad y'(0) &= 0
\end{align*}
\]

M. Tank 1 initially contains 50 gals of water with 10 oz of salt in it, while Tank 2 initially contains 20 gals of water with 15 oz of salt in it. Water containing 2 oz/gal of salt flows into Tank 1 at a rate of 5 gal/min and the well-stirred mixture flows from Tank 1 into Tank 2 at the same rate of 5 gal/min. The solution in Tank 2 flows out to the ground at a rate of 5 gal/min. If $x_1(t)$ and $x_2(t)$ represent the number of ounces of salt in Tank 1 and Tank 2, respectively, **set up but do not solve** an initial value problem describing this system.
\(N.\) If \(x^{(1)}(t)\) and \(x^{(2)}(t)\) are linearly independent solutions to the \(2 \times 2\) system \(x' = Ax\), then the matrix \(\Phi(t) = (x^{(1)}(t), x^{(2)}(t))\) is called a Fundamental Matrix for the system. Find a Fundamental Matrix \(\Phi(t)\) of the system \(x' = \begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix} x\).

\(O.\) Laplace transforms may be used to find solutions to some linear systems of differential equations. Consider the linear system of differential equations:

\[
\begin{align*}
x' &= x + y \\
y' &= 4x + y
\end{align*}
\]

with initial conditions \(x(0) = 0\) and \(y(0) = 2\).

(a) Let \(X(s) = \mathcal{L}\{x(t)\}\) and \(Y(s) = \mathcal{L}\{y(t)\}\) be the Laplace transforms of the functions \(x(t)\) and \(y(t)\), respectively. Take the Laplace transform of each of the differential equations in (\(\ast\)) and solve for \(X(s)\) (i.e., eliminate \(Y(s)\)).

(b) Using the function \(X(s)\) from (a), determine \(x(t)\).

(c) Use the expression for \(x(t)\) and the first equation in (\(\ast\)) to determine \(y(t)\).

\(P.\) Find a particular solution \(x_p(t)\) of these nonhomogeneous systems:

(a) \(x' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 5e^{2t} \\ 3 \end{pmatrix}\)

(b) \(x' = \begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 10 \cos t \\ 0 \end{pmatrix}\)