MA 137 Lesson 5 Recap

We began by going over how an alien with five fingers might write down a number. We did the first basic activity of Berkman's paper "Exploring Interplanetary Algebra to Understand Earthly Mathematics."

We saw that if an alien had five fingers, she would probably group things in fives (like we group things in tens since we have ten fingers!).

Consider the picture below.

An alien with four fingers would probably group things in fours, like we do in tens.

Seeing three whole groups of four and one star left over, the alien with four fingers might write this number as "31" even though we would write it as "13."

An alien with three fingers would probably group things in threes, like we do in tens.

Seeing one whole group of three groups of three, one whole group of three, and one left over, the alien with three fingers might write this number as "111" even though we would write it as "13."

The alien with four fingers is using a base four number system, and the alien with three fingers is using a base three number system. Even though they are written differently, they all represent the same number!

$$13_{\text{ten}} = 31_{\text{four}} = 111_{\text{three}}$$

The numbers above would be pronounced "thirteen" (or "one three base ten"), "three one base four," and "one one one base three."

We write the name of the base as a subscript to avoid confusion when working with multiple bases.

We then compared and contrasted how base five works with how base ten works. Below, I am giving you the setup for base ten, base seven, and an arbitrary base b.

base ten:	thousands 10^3	hundreds 10^2	tens 10^1	ones $10^0 = 1$	•	tenths $10^{-1} = \frac{1}{10}$	hundredths $10^{-2} = \frac{1}{100}$
base seven:	? 7^3	forty-nines 7^2	sevens 7^1	ones $7^0 = 1$	•	sevenths $7^{-1} = \frac{1}{7}$	forty-ninths $7^{-2} = \frac{1}{49}$
base b:	? b^3	? b^2	? b^1	ones $b^0 = 1$	•	? $b^{-1} = \frac{1}{b}$	? $b^{-2} = \frac{1}{b^2}$

Using this, we can convert from other bases back into base ten easily.

In base ten, the number $234.61_{\text{ten}} = (2 \times 10^2) + (3 \times 10^1) + (4 \times 1) + \left(6 \times \frac{1}{10}\right) + \left(1 \times \frac{1}{100}\right)$.

We can use the same idea above to convert to base ten. For example, given 12041.36_{nine}, we have:

$$\left(1 \times 9^4\right) + \left(2 \times 9^3\right) + \left(0 \times 9^2\right) + \left(4 \times 9^1\right) + (1 \times 1) + \left(3 \times \frac{1}{9}\right) + \left(6 \times \frac{1}{81}\right)$$

$$= 6561 + 1458 + 0 + 36 + 1 + \frac{1}{3} + \frac{2}{27}$$

$$= 8056\,\frac{11}{27}$$

So $12041.36_{\text{nine}} = 8056\,\frac{11}{27}\,_{\text{ten}}$.

Even though we could convert from base ten to another base by drawing a picture and grouping things together, drawing pictures can take a long time and keeping track of groups, groups of groups, groups of groups of groups, etc. can be difficult when working with larger numbers. So it is a good idea to use our knowledge of how different base number systems work to convert large numbers in base ten to other bases.

As an example, let's convert 256_{ten} into base three.

First, we look at powers of three, and try to find the highest power of three which goes into 256.

$$3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, 3^6 = 729$$

We see that 3^6 is too big. But 3^5 goes into 256. In fact, it goes in **one** time.

$256 - 243 = 13$, so after getting a group of 243, we have 13 left over.

No groups of 81 go into 13. No groups of 27 go into 13.

But 9 goes into 13. In fact, it goes in **one** time.

$13 - 9 = 4$, so after getting a group of 9, we have 4 left over.

3 goes into 4 **one** time, and we have $4 - 3 = 1$ left over.

Seeing all this, we can write the following:

$$256 = (1 \times 243) + (0 \times 81) + (0 \times 27) + (1 \times 9) + (1 \times 3) + (1 \times 1)$$

$$= \left(1 \times 3^5\right) + \left(0 \times 3^4\right) + \left(0 \times 3^3\right) + \left(1 \times 3^2\right) + \left(1 \times 3^1\right) + \left(1 \times 3^0\right)$$

So we have $256_{\text{ten}} = 100111_{\text{three}}$.

It would have taken a very long time to draw two hundred fifty-six stars, and since we have a fifth power of three, we'd have a group of three groups of three groups of three groups of three! That would be very tricky to keep track of.

Lastly, in a base b number system, there are exactly b numerals: $0, 1, 2, ..., b-2, b-1$.

So for example, in base six, we would count in the following way (all numbers are in base six):

$$0, 1, 2, 3, 4, 5, 10, 11, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25,$$
$$30, 31, 32, 33, 34, 35, 40, 41, 42, 43, 44, 45, 50, 51, 52, 53, 54, 55,$$
$$100, 101, 102, ...$$

The reason for this is we are grouping in terms of powers of six. We won't ever see the numeral 6 in base six, because 6 is a whole group of six with nothing left over, and so is written "10."

For the same reason, we do not see numerals larger than 6 either. They all have groups of six within them.

Although we didn't cover it today, if we were using a base higher than ten, we would have to invent new symbols. In computer science, hexadecimal (base sixteen) is frequently used. They use capital letters for the additional numerals. So, in hexadecimal, one would count like this (all numbers are in hexadecimal):

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F,$$
$$10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E, 1F,$$
$$20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 2A, 2B, 2C, 2D, 2E, 2F,$$
$$...$$
$$90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 9A, 9B, 9C, 9D, 9E, 9F,$$
$$A0, A1, A2, A3, A4, A5, A6, A7, A8, A9, AA, AB, AC, AD, AE, AF,$$
$$...$$

You do not have to know too much about bases higher than ten, other than the material we covered for bases lower than ten (any problem you will see this semester with a base higher than ten will use numerals up to and including 9, but never anything higher). I just thought it would be fun to point out this fact about a base other than ten which is commonly used today.