SEQUENCES AND SERIES
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1. Introduction

We collect the main results about convergence and divergence of sequences and series studied in MA162, and provide examples of the application of convergence tests.

2. Sequences

A sequence is a list of numbers in a particular order

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}.$$

One can also think of a sequence as a function which takes a positive integer n to a real number a_n, or in other words, if $\mathbb{N}$ denotes the positive integers and $\mathbb{R}$ the real numbers a sequence is a function

$$f : \mathbb{N} \longrightarrow \mathbb{R}$$
$$n \longmapsto a_n$$

The main topic of discussion: How does a sequence behave as $n \to \infty$? There are three possibilities:

I. **The sequence convergences to a number** L : (Intuitive definition) One says that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a real number L, and in this case we say that $\lim_{n\to\infty} a_n = L$, if $|a_n - L|$ is arbitrarily small as $n \to \infty$.
Example: It is intuitive that when n gets large, $\frac{1}{n}$ gets small and $\lim_{n\to\infty} \frac{1}{n} = 0$.

II. **The sequence does not converge:** Here is an example: The sequence $a_n = (-1)^n$ does not converge. When n is even $a_n = 1$ and when n is odd $a_n = -1$, so half of he sequence approaches 1 (and in fact is equal to one) and the other half approaches -1 (in fact it is equal to -1). Therefore the whole sequence does not approach any number when $n \to \infty$.

III. **The limit of the sequence is equal to infinity or negative infinity:** (Intuitive definition)
We say that $\lim_{n\to\infty} a_n = \infty$ **if for any** $M > 0$, $a_n > M$ **for** n **large enough.**
We say that $\lim_{n\to\infty} a_n = -\infty$ **if for any** $M < 0$, $a_n < M$ **for** n **large enough.**
The following sequences have limit equal to infinity or negative infinity:

1. $\lim_{n\to\infty} n = \infty$,
2. $\lim_{n\to\infty} \ln n = \infty$,
3. $\lim_{n\to\infty} \ln(\frac{1}{n}) = -\infty$
4. $\lim_{n\to\infty} e^n = \infty$

2.1. **Converging Sequences:** The following results will be the source of most of our examples of converging sequences:
Result 1: Let $f(x)$ be a function defined on an open interval containing L such that $\lim_{x\to L} f(x) = M$. Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = M$. In particular, when $f(x)$ is continuous at L, $f(L) = M$ and

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(L) = M.$$

Examples:

1. The function $\sin x$ satisfies $\lim_{x\to 0} \sin x = 0$ and since $\lim_{n\to\infty} \frac{1}{n} = 0$, then

$$\lim_{n\to\infty} \sin(\frac{1}{n}) = 0.$$

2. $f(x) = x^k$, $k > 0$ satisfies $\lim_{x\to 0} x^k = 0$. Therefore $\lim_{n\to\infty} \frac{1}{n^k} = 0$.

3. Similarly, $f(x) = 2^x$ is continuous and $\lim_{x\to 0} 2^x = 2^0 = 1$. Therefore

$$\lim_{n\to\infty} 2^{\frac{1}{n}} = 2^0 = 1.$$

4. Compute $\lim_{n\to\infty} \frac{n^2+3}{2n^2+8}$. In this case we write:

$$\frac{n^2+3}{2n^2+8} = \frac{n^2(1+\frac{3}{n^2})}{n^2(2+\frac{8}{n^2})} = \frac{(1+\frac{3}{n^2})}{(2+\frac{8}{n^2})},$$

$$\text{therefore } \lim_{n\to\infty} \frac{n^2+3}{2n^2+8} = \lim_{n\to\infty} \frac{(1+\frac{3}{n^2})}{(2+\frac{8}{n^2})}.$$

$$\text{since } \lim_{n\to\infty} \frac{1}{n^2} = 0,$$

$$\lim_{n\to\infty} \frac{n^2+3}{2n^2+8} = \lim_{n\to\infty} \frac{(1+\frac{3}{n^2})}{(2+\frac{8}{n^2})} = \frac{1}{2}.$$

Here we used result 1 for the function $f(x) = \frac{1+3x}{2+8x}$. In general, using the same idea, if $p > 0$

$$\lim_{n\to\infty} \frac{an^p + \text{ lower powers of n}}{bn^p + \text{lower powers of n}} = \frac{a}{b}$$
$$\lim_{n\to\infty} \frac{an^p + \text{ lower powers of n}}{bn^q + \text{lower powers of n}} = 0 \text{ if } q > p > 0,$$

5. Compute $\lim_{n\to\infty} \dfrac{1-\cos(\frac{1}{n})}{\frac{1}{n^2}}$. We first identify this limit as $\lim_{n\to\infty} f(\dfrac{1}{n})$ where $f(x) = \dfrac{1-\cos(x)}{x^2}$ and we need to compute the limit $\lim_{x\to 0} f(x)$. Here we are computing the limit of a function and since $\lim_{x\to 0}(1-\cos(x)) = 0$ and $\lim_{x\to 0} x^2 = 0$, L'Hopital's rule says that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \frac{1}{2}.$$

Conclusion: $\lim_{n\to\infty} \dfrac{1-\cos(\frac{1}{n})}{\frac{1}{n^2}} = \dfrac{1}{2}$.

6. Compute $\lim_{n\to\infty} \dfrac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n^3}}$. One can also think of $x = \frac{1}{n}$ to reduce the limit to $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$ and since $\lim_{x\to 0}(x-\sin x) = \lim_{x\to 0} x^3 = 0$ we can use L'Hopital's Rule to compute it

$$\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{\frac{d}{dx}(x-\sin x)}{\frac{d}{dx}x^3} = \lim_{x\to 0} \frac{(1-\cos x)}{3x^2} = \frac{1}{3}\lim_{x\to 0} \frac{(1-\cos x)}{x^2} = \frac{1}{6}$$

here we used the limit we just computed above.

Conclusion: $\lim_{n\to\infty} \dfrac{\frac{1}{n}-\sin(\frac{1}{n})}{\frac{1}{n^3}} = \dfrac{1}{6}$.

7. Compute $\lim_{n\to\infty} \dfrac{2^{\frac{1}{n}}-1}{\frac{1}{n}}$. Again, we need to recognize that this is equal to compute $\lim_{n\to\infty} f(\frac{1}{n})$ where $f(x) = \dfrac{2^x-1}{x}$ and so we have to compute

$$\lim_{x\to 0} \frac{2^x-1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(2^x-1)}{\frac{d}{dx}x} = \lim_{x\to 0} 2^x \ln 2 = \ln 2,$$

Here we used that $\dfrac{d}{dx}2^x = 2^x \ln 2$.

Conclusion: $\lim_{n\to\infty} \dfrac{2^{\frac{1}{n}}-1}{\frac{1}{n}} = \ln 2$.

The second source of examples of converging sequences is the following:
Result 2: If $f(x)$ is a function defined on $(0, \infty)$, and if $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} f(n) = L$. Notice the latter limit is a special case of the first.

We can apply these results to see that

$$\lim_{n\to\infty} \arctan(n) = \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$$
$$\lim_{n\to\infty} e^{-n} = \lim_{x\to\infty} e^{-x} = 0$$
$$\lim_{n\to\infty} 3^{-n^2} = \lim_{x\to\infty} 3^{-x^2} = 0$$
$$\lim_{n\to\infty} r^n = \lim_{x\to\infty} r^x = 0 \text{ if } r < 1.$$

The good thing about these results is that we can use techniques to compute the limits of functions.
Examples:

1. Compute $\lim_{n\to\infty} \frac{\ln n}{n}$. Instead of working with n, which only takes integer values, we compute $\lim_{x\to\infty} \frac{\ln x}{x}$. We can either replace n by x or we can just think of n as a continuous variable, in this case since $\lim_{n\to\infty} \ln n = \infty$ and $\lim_{n\to\infty} n = \infty$, L'Hopital's rule says that (and here we are thinking of n as a continuous variable)

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn}n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0.$$

Conclusion: $\lim_{n\to\infty} \frac{\ln n}{n} = 0.$

2. Compute $\lim_{n\to\infty} \frac{(\ln n)^2}{\sqrt{n}}$. Again we think of n as a continuous variable and use L'Hopital's rule

$$\lim_{n\to\infty} \frac{(\ln n)^2}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln n)^2}{\frac{d}{dn}\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{2}{n}(\ln n)}{\frac{1}{2\sqrt{n}}} =$$
$$\lim_{n\to\infty} \frac{4\ln n}{\sqrt{n}} = 4\lim_{n\to\infty} \frac{\frac{d}{dn}\ln n}{\frac{d}{dn}\sqrt{n}} = 4\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = 8\lim_{n\to\infty} \frac{1}{n^{\frac{3}{2}}} = 0$$

Conclusion: $\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 0.$

3. Compute $\lim_{n\to\infty} n^{\frac{1}{n}}$. As above, the idea is to think of n as a continuous variable, or replace $x = n$. So we will compute $\lim_{x\to\infty} x^{\frac{1}{x}}$. This requires a new idea which is to use that

$$x^{\frac{1}{x}} = e^{\ln(x^{\frac{1}{x}})},$$

and since we know that the exponential function is continuous

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln(x^{\frac{1}{x}})} = e^{\lim_{x\to\infty}\ln(x^{\frac{1}{x}})}.$$

So we need to compute $\lim_{x\to\infty} \ln(x^{\frac{1}{x}})$. But

$$\ln(x^{\frac{1}{x}}) = \frac{1}{x}\ln x,$$

and we just learned that

$$\lim_{x\to\infty} \frac{1}{x}\ln x = 0.$$

Conclusion: $\lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\ln(x^{\frac{1}{x}})} = e^{\lim_{x\to\infty}\ln(x^{\frac{1}{x}})} = e^0 = 1.$

4. Compute $\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n$. Again, the idea is to use that

$$\left(1+\frac{a}{n}\right)^n = e^{\ln\left(\left(1+\frac{a}{n}\right)^n\right)},$$

and the fact that the exponential function is continuous, so

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{a}{n}\right)^n\right)} = e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{a}{n}\right)^n\right)}$$

So we need to compute the following (and we treat n as x and allow ourselves to take derivatives)

$$\lim_{n\to\infty}\ln\left(\left(1+\frac{a}{n}\right)^n\right) = \lim_{n\to\infty} n\ln\left(1+\frac{a}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\frac{a}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty}\frac{\frac{d}{dn}\ln\left(1+\frac{a}{n}\right)}{\frac{d}{dn}\frac{1}{n}} =$$

$$\lim_{n\to\infty}\frac{-\frac{a}{n^2}\left(1+\frac{a}{n}\right)^{-1}}{-\frac{1}{n^2}} = a\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{-1} = a$$

Conclusion: $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(\left(1+\frac{a}{n}\right)^n\right)} = e^{\lim_{n\to\infty}\ln\left(\left(1+\frac{a}{n}\right)^n\right)} = e^a.$

5. Compute $\lim_{n\to\infty}\left(\frac{n^2+3n+4}{n+10} - n\right)$. Here each term goes to infinity, and we cannot say that $\lim_{n\to\infty}\left(\frac{n^2+3n+4}{n+10} - n\right) = \infty - \infty$. This does not make sense. Here is what we have

to do:

$$\lim_{n\to\infty}\left(\frac{n^2+3n+4}{n+10}-n\right)=\lim_{n\to\infty}\frac{(n^2+3n+4)-n(n+10)}{n+10}=$$
$$\lim_{n\to\infty}\frac{n^2+3n+4-n^2-10n}{n+10}=\lim_{n\to\infty}\frac{-7n+4}{n+10}=-7.$$

6. Compute $\lim_{n\to\infty}\left(\ln(n^2+10n+8)-\ln(n^2+10)\right)$. Here we have to realize that

$$\lim_{n\to\infty}\left(\ln(2n^2+10n+8)-\ln(n^2+10)\right)=\lim_{n\to\infty}\ln(\frac{2n^2+10n+8}{n^2+10})=\lim_{n\to\infty}\ln(\frac{2+\frac{10}{n}+\frac{8}{n^2}}{1+\frac{10}{n^2}})=\ln 2.$$

7. Another source of examples are sequences involving factorials: $n!=n(n-1)(n-2)...1$, is the product of all positive integers from 1 to n. Compute $\lim_{n\to\infty}\frac{(n+1)!}{(n+2)!}$. Just notice that $(n+2)!=(n+2)(n+1)!$ and therefore

$$\frac{(n+1)!}{(n+2)!}=\frac{(n+1)!}{(n+2)(n+1)!}=\frac{1}{n+2},\text{ so}$$
$$\lim_{n\to\infty}\frac{(n+1)!}{(n+2)!}=\lim_{n\to\infty}\frac{1}{n+2}=0.$$

Review exercises from the textbook: Page 704. Problems 23 to 53, 64, 72, 73, 79,

3. Series

We begin with a sequence $\{a_1,a_2,a_3,\dots a_n\dots\}=\{a_n\}_{n=1}^\infty$, and form a new sequence:

$$s_1=a_1$$
$$s_2=a_1+a_2=\sum_{j=1}^{2}a_j$$
$$s_3=a_1+a_2+a_3=\sum_{j=1}^{3}a_j$$
$$\dots$$
$$s_n=a_1+a_2+a_3+\dots a_n=\sum_{j=1}^{n}a_j$$
$$\dots$$

s_n is the sum of the first n-terms of the sequence $\{a_n\}_{n=1}^\infty$. Now we want to understand he limit of the sequence s_n as $n\to\infty$.

$$s=\lim_{n\to\infty}\sum_{j=1}^{n}a_j=\sum_{j=1}^{\infty}a_j \text{ is called an infinite series}$$

If the limit exists and is finite, we say that the series $\sum_{j=1}^{\infty} a_j$ converges. In general it is quite hard to decide if a series converges. If you are asked if a given series converges, here is the first thing you need to do:

Test for divergence: If $\lim_{n\to\infty} a_n$ does not exist, or if $\lim_{n\to\infty} a_n = L$, but $L \neq 0$, the series $\sum_{j=1}^{\infty} a_j$ diverges.

Remark: If $\lim_{n\to\infty} a_n = 0$ **the test does not say that the series converges.** It only says that the series diverges if the limit is not equal to zero, or if it does not exist.

Using this test it is easy to see that the following series diverge:

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n$ diverges because as discussed above the sequence $a_n = (-1)^n$ does not converge.
2. $\displaystyle\sum_{n=1}^{\infty}\cos(\frac{1}{n})$ diverges because $\displaystyle\lim_{n\to\infty}\cos(1/n) = 1$
3. $\displaystyle\sum_{n=1}^{\infty} e^{\frac{1}{n}}$ also diverges because $\displaystyle\lim_{n\to\infty} e^{1/n} = 1$
4. $\displaystyle\sum_{n=1}^{\infty} r^n$, if $r > 1$ diverges because $\lim_{n\to\infty} r^n$ does not exist if $r < -1$. We see that when $r < -1$, r^n goes to ∞ when n is even and r^n goes to $-\infty$ when n is odd.

Again, this is a very limited test, which only serves to show that a series diverge. From now on, we will always have $\lim_{n\to\infty} a_n = 0$ and we want to analyze the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$.

3.1. **Geometric series:** These are series of the form $\displaystyle\sum_{n=1}^{\infty} r^n$, with $|r| < 1$. One can actually compute the sum of such series. If

$$S_N = \sum_{n=1}^{N} r^n = r + r^2 + r^3 + \ldots + r^{N-1} + r^N, \text{ then}$$
$$rS_N = r^2 + r^3 + \ldots + r^N + r^{N+1}$$

and so

$$S_N - rS_N = (1-r)S_N = r - r^{N+1}, \text{ so}$$
$$S_N = \frac{r - r^{N+1}}{1-r}$$

Since $|r| < 1$, $\lim_{N\to\infty} r^{N+1} = 0$, and hence

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \sum_{n=1}^{N} r^n = \sum_{n=1}^{\infty} r^n = \frac{1}{1-r}.$$

Variations of this formula:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots\dots = 1 + \sum_{n=1}^{\infty} r^n = 1 + \frac{r}{1-r} = \frac{1}{1-r}$$

$$\sum_{n=k}^{\infty} r^n = r^k + r^{k+1} + r^{k+2} + \dots\dots = r^k(1 + r + r^2 + \dots) = r^k \sum_{n=0}^{\infty} r^n = r^k \frac{1}{1-r} = \frac{r^k}{1-r}.$$

Examples: Compute the sum of the following series:

1. $\sum_{n=3}^{\infty} \frac{3^n}{4^n}$

   We need to recognize that this series is equal to $\sum_{n=3}^{\infty} \frac{3^n}{4^n} = \sum_{n=3}^{\infty} (\frac{3}{4})^n = \frac{(\frac{3}{4})^3}{1-\frac{3}{4}} = \frac{(\frac{3}{4})^3}{\frac{1}{4}} = \frac{81}{16}.$

2. $\sum_{n=2}^{\infty} \frac{1+2^n}{4^n}$

   This is not really a geometric series, but it can be split into two geometric series

$$\sum_{n=2}^{\infty} \frac{1+2^n}{4^n} = \sum_{n=2}^{\infty} \frac{1}{4^n} + \sum_{n=2}^{\infty} \frac{2^n}{4^n} = \sum_{n=2}^{\infty} (\frac{1}{4})^n + \sum_{n=2}^{\infty} (\frac{2}{4})^n =$$
$$\frac{(\frac{1}{4})^2}{1-\frac{1}{4}} + \frac{(\frac{1}{2})^2}{1-\frac{1}{2}} = \frac{1}{12} + \frac{1}{2} = \frac{7}{12}.$$

3.2. **Convergence tests:** We will analyze he convergence of more complicated series. First we deal with series of non-negative terms and the first test of convergence is the integral test:

The integral test: Let $f(x)$ be a continuous function defined on $[1, \infty)$ and suppose that
i) $f(x) > 0$
ii) $f(x)$ is decreasing
iii) $\lim_{x\to\infty} f(x) = 0$,

Then the series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_1^{\infty} f(x)\, dx$ converges. In other words, if one converges the other one also converges, if one diverges so does the other one.

Observation for those who read the textbook closely: The statement of this result in the textbook does not include condition (iii). We know that if (iii) fails the series diverges, and it is easier to check (iii) than to compute an integral to see that the series diverges. Notice however that that conditions i) and ii) imply that $\lim_{x\to\infty} f(x)$ exists. But, since $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} f(n)$, if $\lim_{n\to\infty} f(x) \neq 0$, $\sum_{n=1}^{\infty} f(n)$ diverges and so does the integral $\int_1^\infty f(x)\,dx$.

With this test we can analyze the convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p},\ p>0$$
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p},\ p>0$$
$$\sum_{n=1}^{\infty} e^{-n}$$
$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$$

In the first case $f(x)=\frac{1}{x^p}$. The case $p=1$, $f(x)=\frac{1}{x}$ satisfies the conditions i, ii and iii of the theorem. So we need to analyze the integral

$$\int_1^\infty \frac{1}{x}dx = \ln x|_1^\infty = \lim_{x\to\infty} \ln x = \infty. \text{ it diverges}$$

So $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

When $p<1$

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\ x^{1-p}\Big|_1^\infty = \frac{1}{1-p}(-1+\lim_{x\to\infty} x^{1-p}) = \infty. \text{ it diverges}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^p}$ also diverges when $p<1$.

However, when $p>1$,

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{1-p}\ x^{1-p}\Big|_1^\infty = \frac{1}{1-p}(-1+\lim_{x\to\infty} x^{1-p}) = \frac{1}{p-1}. \text{ it converges}$$

Conclusion: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ **converges when** $p>1$ **and diverges for** $p \leq 1$.

The second example is similar because if one sets $u=\ln x$, then

$$\int_2^\infty \frac{1}{x(\ln x)^p}dx = \int_{\ln 2}^\infty \frac{du}{u^p},$$

which diverges when $p \leq 1$ and converges when $p > 1$.

Conclusion: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ **converges when** $p > 1$ **and diverges for** $p \leq 1$.

Since $\int_1^\infty e^{-x}\, dx = e^{-1}$, the series $\sum_{n=1}^\infty e^{-n}$ converges.

In the last example

$$\int_1^\infty \frac{1}{1+x^2}dx = \arctan x|_1^\infty = \frac{\pi}{2} - 1. \text{ So the series converges.}$$

Now we have a small collection of series that we know converge or diverge. The next tests are used to compare two series and use the convergence or the divergence of one of them to analyze the convergence or divergence of the other.

The comparison test: Let $\{a_n\}$ and $\{b_n\}$ be two sequences with $0 \leq a_n \leq b_n$ for n large

If $\sum_{n=1}^{\infty} b_n$ converges, then the smaller one also converges, i.e $\sum_{n=1}^{\infty} a_n$ converges

If $\sum_{n=1}^{\infty} a_n$ diverges, then the bigger one also diverges, i.e $\sum_{n=1}^{\infty} b_n$ diverges .

Remark: The books states that $a_n > 0$, but of course this is not necessary. We only need $a_n \geq 0$.

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{10n+5}$. Notice that

$$10n + 5 < 20n, \text{ for } n = 1, 2, 3, \text{ therefore}$$
$$\frac{1}{10n+5} > \frac{1}{20n}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{20n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{10n+5}$ also diverges.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+6n+2}$. Here we have

$$n^2+6n+2 > n^2, \text{ therefore}$$
$$\frac{1}{n^2+6n+2} < \frac{1}{n^2}$$

Conclusion: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+6n+2}$ converges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos 3n|}{n^3}$. Since $|\cos\theta| \leq 1$ for any θ, $\dfrac{|\cos 3n|}{n^3} \leq \dfrac{1}{n^3}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos 3n|}{n^3}$ also converges.

4. Let $f(x)$ be a function defined on $[1,\infty)$ such that $5x \leq f(x) \leq 10x^2$. What can be said about the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^4+3}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^2+7}$?

Since $f(x) \leq 10x^2$, it follows that $\dfrac{f(n)}{n^4+3} \leq 10\dfrac{n^2}{n^4+3} \leq \dfrac{1}{n^2}$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^4+3}$. On the other hand, since $f(x) \geq 5x$, it follows that $\dfrac{f(n)}{n^2+3} \geq \dfrac{5n}{n^2+3}$. But $n^2+3 \leq 10n^2$ and therefore $\dfrac{f(n)}{n^2+3} \geq \dfrac{5n}{n^2+3} \geq \dfrac{5n}{10n^2} = \dfrac{1}{2n}$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{f(n)}{n^2+3}$.

The following is a better test says that all you need to worry about is the behavior of the terms of the series at infinity.

The limit comparison test: Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that nor large n, $a_n > 0$ and $b_n > 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0.$$

Then either both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge. So if we know one of them converges, the other cone also converges. If one diverges, so does the other one.
Examples: Use the limit comparison theorem to analyze the convergence of the following series

1. $\displaystyle\sum_{n=1}^{\infty}\frac{6n^2+8n+4}{n^3+12}$. Here is how one should apply this test: Notice that for n very large $\displaystyle\frac{6n^2+8n+4}{n^3+12}\sim\frac{6n^2}{n^3}=\frac{6}{n}$, and since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, then $\displaystyle\sum_{n=1}^{\infty}\frac{6n^2+8n+4}{n^3+12}$ also diverges. One can more formally use the theorem by showing that

$$\lim_{n\to\infty}\frac{\frac{6n^2+8n+4}{n^3+12}}{\frac{1}{n}}=6\neq 0,$$

and thus $\displaystyle\sum_{n=1}^{\infty}\frac{6n^2+8n+4}{n^3+12}$ diverges because $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

2. For what values of $p>0$ does the series $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n^4+3n}{n^p+2}}$ converge? The point here is is to write

$$\sqrt{\frac{n^4+3n}{n^p+2}}=\sqrt{\frac{n^4(1+\frac{3}{n^3})}{n^p(1+\frac{2}{n^p})}}=\frac{1}{n^{\frac{p-4}{2}}}\sqrt{\frac{1+\frac{3}{n^3}}{1+\frac{2}{n^p}}}.$$

Therefore, for large n, $\displaystyle\sqrt{\frac{n^4+3n}{n^p+2}}\sim\frac{1}{n^{\frac{p-4}{2}}}$ and therefore both series $\displaystyle\sum_{n=1}^{\infty}\sqrt{\frac{n^4+3n}{n^p+2}}$ and $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{\frac{p-4}{2}}}$ converge for the same values of p, which in this case is $\displaystyle\frac{p-4}{2}>1$, or $p>6$.

One can more precisely state that

$$\lim_{n\to\infty}\frac{\sqrt{\frac{n^4+3n}{n^p+2}}}{\frac{1}{n^{\frac{p-4}{2}}}}=\lim_{n\to\infty}\sqrt{\frac{1+\frac{3}{n^3}}{1+\frac{2}{n^p}}}=1.$$

and use the limit comparison test as stated above.

3. $\displaystyle\sum_{n=1}^{\infty}\sin(\frac{1}{n})$. We know that

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

this means that for x small $\sin x\sim x$. So for n large, $\sin(\frac{1}{n})\sim\frac{1}{n}$ and therefore $\sum_{n=1}^{\infty}\sin(\frac{1}{n})$ diverges because $\sum_{n=1}^{\infty}\frac{1}{n}$ does. As above, one could apply this more formally if we think of $\frac{1}{n}$ as x and in this case

$$\lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{\frac{1}{n}}=\lim_{x\to 0}\frac{\sin x}{x}=1.$$

So, $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ diverges because $\sum_{n=1}^{\infty} \frac{1}{n}$ does.

4. For what values of $p > 0$ does the series $\sum_{n=1}^{\infty} \sin(\frac{1}{n^p})$ converge? As we saw above, when x is small $\sin x \sim x$, and so for large n, $\sin(\frac{1}{n^p}) \sim \frac{1}{n^p}$ and we conclude that
$\sum_{n=1}^{\infty} \sin(\frac{1}{n^p})$ diverges when $p \leq 1$ because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $p \leq 1$.
$\sum_{n=1}^{\infty} \sin(\frac{1}{n^p})$ converges when $p > 1$ because $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

5. Suppose $f(x)$ is such that $\lim_{x\to 0} \frac{f(x)}{x} = L > 0$. What can be said about the convergence of the series $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ for $p > 0$? Notice that this assumption implies that $f(x) > 0$ for $x > 0$ small, and so $f(\frac{1}{n^p}) > 0$ for n large (this is important because it is a requirement of the limit comparison test). Loosely speaking, since $\lim_{x\to 0} \frac{f(x)}{x} = L > 0$, $f(x) \sim Lx$ when x is small, then $f(\frac{1}{n^p}) \sim \frac{L}{n^p}$. So $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ diverges, which is for $p \leq 1$, and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ converges, which is for $p > 1$.
   We can make this very precise if we recognize that

$$\lim_{n\to\infty} \frac{f(\frac{1}{n^p})}{\frac{1}{n^p}} = \lim_{x\to 0} \frac{f(x)}{x} = L.$$

Therefore we conclude that both series $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge or diverge for exactly the same values of p.

6. Suppose $f(x)$ is such that $\lim_{x\to 0} \frac{f(x)}{x^2} = 3$. What can be said about the convergence of the series $\sum_{n=1}^{\infty} f(\frac{1}{n})$? Again, the idea is that since $\lim_{x\to 0} \frac{f(x)}{x^2} = 3$, then $f(x) \sim 3x^2$ as $x \to 0$, and so $f(\frac{1}{n}) \sim \frac{3}{n^2}$, and since $\sum_{n=1}^{\infty} f(\frac{1}{n^2})$ converges, $\sum_{n=1}^{\infty} f(\frac{1}{n})$ converges. A more precise

way of saying this is to recognize that

$$\lim_{n\to\infty} \frac{f(\frac{1}{n})}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{f(x)}{x^2} = 1$$

and so $\sum_{n=1}^{\infty} f(\frac{1}{n})$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

7. Suppose $f(x)$ is such that $\lim_{x\to 0} \frac{f(x)}{x^3} = 1$. For what values of $p > 0$ does the series $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ converge?
If $\lim_{x\to 0} \frac{f(x)}{x^3} = 1$, then $f(x) \sim x^3$ as $x \to 0$, (in particular this implies that $f(x) > 0$ for $x > 0$ small) and so $f(\frac{1}{n^p}) \sim \frac{1}{n^{3p}}$, (and positive for n large) and since $\sum_{n=1}^{\infty} \frac{1}{n^{3p}}$ only converges if $3p > 1$, $\sum_{n=1}^{\infty} f(\frac{1}{n^p})$ converges only when $3p > 1$.

8. $\sum_{n=1}^{\infty} (\frac{1}{n} - \sin(\frac{1}{n}))$. The series $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge, so one cannot subtract them. The point is to understand how the difference $(\frac{1}{n} - \sin(\frac{1}{n}))$ behaves for n large. If you recall, we saw above that

$$\lim_{n\to\infty} \frac{\frac{1}{n} - \sin(\frac{1}{n})}{\frac{1}{n^3}} = \frac{1}{6}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, the limit comparison theorem says that $\sum_{n=1}^{\infty} (\frac{1}{n} - \sin(\frac{1}{n}))$ converges.

9. $\sum_{n=1}^{\infty} (1 - \cos(\frac{1}{n}))$. Similarly, we saw above that

$$\lim_{n\to\infty} \frac{1 - \cos(\frac{1}{n})}{\frac{1}{n^2}} = \frac{1}{2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the limit comparison theorem says that $\sum_{n=1}^{\infty} (1 - \cos(\frac{1}{n}))$ converges.

10. For what values of $p > 0$ does $\sum_{n=1}^{\infty}(2^{\frac{1}{n^p}} - 1)$ converge? If we think of $x = \frac{1}{n^p}$, when $n \to \infty$ $x \to 0$, we obtain

$$\lim_{n\to\infty} \frac{2^{\frac{1}{n^p}} - 1}{\frac{1}{n^p}} = \lim_{x\to 0} \frac{2^x - 1}{x} = \ln 2 \neq 0 \text{ we computed this limit above}$$

Conclusion: The series $\sum_{n=1}^{\infty}(2^{\frac{1}{n^p}} - 1)$ diverges for the same values of $p > 0$ for which $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, which are $p \leq 1$. The series $\sum_{n=1}^{\infty}(2^{\frac{1}{n^p}} - 1)$ converges for the same values of $p > 0$ for which $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges, which are $p > 1$.

11. For what values of $p > 0$ does the series $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n^p})$ converge? If we take the limit

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x} = \lim_{x\to 0} \frac{1}{1+x} = 1.$$

Conclusion: $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n^p})$ converges for $p > 1$ and diverges for $p \leq 1$.

12. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$. We compare the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ with $\sum_{n=1}^{\infty} \frac{1}{n}$. Take the limit

$$\lim_{n\to\infty} \frac{\frac{1}{n^{1+\frac{1}{n}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^{1+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1.$$

We computed the last limit in the section above, when we discussed sequences. So both $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge.

13. $\sum_{n=1}^{\infty} \frac{1}{n^{2+\frac{1}{n}}}$. We compare the series $\sum_{n=1}^{\infty} \frac{1}{n^{2+\frac{1}{n}}}$ with $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Take the limit

$$\lim_{n\to\infty} \frac{\frac{1}{n^{2+\frac{1}{n}}}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^{2+\frac{1}{n}}} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1.$$

So both $\sum_{n=1}^{\infty} \frac{1}{n^{2+\frac{1}{n}}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge.

Review exercises from the textbook: Page 726: Problems 3 to 26, 30 and 31. Page 731. Problems 3 to 32.

3.3. **Alternating series.** So far we have only considered series of positive terms. Next we study a particular case of series which have positive and negative terms, these are called alternating series. These are series of the form $\sum_{n=1}^{\infty}(-1)^n b_n$, where $b_n > 0$. We can say certain particular cases of alternating series converge, and even estimate the sum of he series:

Alternating series test: If the sequence $\{b_n\}$ satisfies:

i) $b_n > 0$, $n = 1, 2, \dots$

ii) $b_n \geq b_{n+1}$, or in other words, the sequence is decreasing

iii) $\lim_{n\to\infty} b_n = 0$,

then the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges. Moreover if $S = \sum_{n=1}^{\infty}(-1)^n b_n$ is the sum of the series and $S_N = \sum_{n=1}^{N}(-1)^n b_n$, is the partial sum of the first N terms, then

$$|S - S_N| \leq b_{N+1}.$$

Verify that the following series satisfy the conditions of the alternating series test and estimate the sum of the series with an error less than or equal to 10^{-5}.

1) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n!}$, here $n! = 1.2.3.4.5...n$ is the product of all number from 1 to n.
2) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^3}$

In case 1, b_n satisfies the conditions of the test. To estimate the sum we use that

$$|S - S_N| \leq b_{N+1} = \frac{1}{(N+1)!}$$

so we want to find N such that $\frac{1}{(N+1)!} < 10^{-5}$, which is the same as $(N+1)! > 10^5$. This is small enough that one can do by trial and error:

If $N = 7$, $(N+1)! = 8! = 8.7.6.5.4.3.1 = 40,320$. Not quite
If $N = 8$, $(N+1)! = 9! = 9.8.7.6.5.4.3.1 = 362,880$ and this works.
So if we add the first 8 terms of he series, we find the sum up to an error which is not greater than 10^{-5} :

$S_8 = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} - \frac{1}{8!}$.

In example 2, $b_n = \frac{1}{n^3}$, which is easily seen to satisfy the three conditions. In this case, we have

$$|S - S_N| \leq b_{N+1} = \frac{1}{(N+1)^3}$$

and so we want $\dfrac{1}{(N+1)^3} < 10^{-5}$ which is the same as $(N+1)^3 \geq 10^5$. So $N + 1 \geq 10^{5/3} \sim 46.41$. So we need $N = 46$. So we have to take the sum of the first 46 terms of the series to obtain an approximation with an error that is less than or equal to 10^{-5}.

Review exercises from the textbook: Page 736: Problems 7 to 20. 23 to 26.

3.4. **Absolute convergence.** We consider a series $\sum_{n=1}^{\infty} a_n$ where the terms are not necessarily positive. We say that $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. It is important to understand the following:

$\sum_{n=1}^{\infty} a_n$ **may converge and** $\sum_{n=1}^{\infty} |a_n|$ **diverge.** Take for example the series $\sum_{n=1}^{\infty} (-1)^n \dfrac{1}{n}$, which is an alternating series with $b_n = \frac{1}{n}$ so it converges. However $\sum_{n=1}^{\infty} \left| (-1)^n \dfrac{1}{n} \right| = \sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges.

However, if $\sum_{n=1}^{\infty} |a_n|$ **converges, then** $\sum_{n=1}^{\infty} a_n$ **converges.** This is not as easy to see, and you can consult your textbook for details.

When the series $\sum_{n=1}^{\infty} a_n$ **converges but** $\sum_{n=1}^{\infty} |a_n|$ **diverges, we say that** $\sum_{n=1}^{\infty} a_n$ **converges conditionally.**

We have two tests for absolute convergence:

The Ratio Test: Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} \left| \dfrac{a_{n+1}}{a_n} \right| = L$. Then

1) If $L < 1$ the series $\sum_{n=1}^{\infty} |a_n|$ converges.

2) If $L > 1$ the series $\sum_{n=1}^{\infty} a_n$ diverges (notice we do not have absolute values here).

3) If $L = 1$ nothing can be said about the convergence of the series.

The Root Test: Let $\{a_n\}$ be a sequence such that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L$. Then

1) If $L < 1$ the series $\sum_{n=1}^{\infty} |a_n|$ converges.

2) If $L > 1$ the series $\sum_{n=1}^{\infty} a_n$ diverges (notice we do not have absolute values here).

3) If $L = 1$ nothing can be said about the convergence of the series.

Use the ratio and root tests to analyze the convergence of the following series:

1. $\sum_{n=1}^{\infty}(-1)^n \frac{10^n}{n!}$. We use the ratio test, which is suitable when we have factorials. The root test does not obviously combine very well with the factorial. In this case, $|a_n| = \frac{10^n}{n!}$ and hence

$$|\frac{a_{n+1}}{a_n}| = \frac{10^{n+1}}{(n+1)!}\frac{n!}{10^n} = \frac{10}{n+1}.$$

Therefore $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 0 < 1$ so the series converges absolutely.

2. $\sum_{n=1}^{\infty}(-1)^n \frac{n^{10}}{2^n}$

We may use either the ratio of the root test. Let's use the ratio test. In this case $|a_n| = \frac{n^{10}}{2^n}$ and hence

$$|\frac{a_{n+1}}{a_n}| = \frac{(n+1)^{10}}{2^{n+1}}\frac{2^n}{n^{10}} = \frac{1}{2}(\frac{n+1}{n})^{10}.$$

Hence $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2} < 1$ so the series converges absolutely.

Notice that $|a_n|^{\frac{1}{n}} = \frac{1}{2}(n^{\frac{1}{n}})^{10}$. Recall that $\lim n^{\frac{1}{n}} = 1$ therefore, $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \frac{1}{2} < 1$ so the series converges absolutely.

3. $\sum_{n=1}^{\infty}(2^n - 1)^n$

Here $a_n = (2^n - 1)^n$ so it's obviously a case for the root test: $a_n^{\frac{1}{n}} = 2^{\frac{1}{n}} - 1$, and so $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty}(2^{\frac{1}{n}} - 1) = 2^0 - 1 = 0 < 1$, so the series converges absolutely.

4. For what values of a does the series $\sum_{n=1}^{\infty}(1+\frac{a}{n})^{n^2}$ converge?

This is again a case for the root test. In this case

$$|a_n|^{\frac{1}{n}} = (1+\frac{a}{n})^n \text{ and we saw above that}$$
$$\lim_{n\to\infty}(1+\frac{a}{n})^n = e^a.$$

So the root test says that the series converges if $e^a < 1$ so $a < 0$ and it diverges if $e^a > 1$ so it diverges for $a > 0$. When $a = 0$, the series $\sum_{n=1}^{\infty}(1+\frac{a}{n})^{n^2} = \sum_{n=1}^{\infty} 1$ which obviously diverges. Conclusion: The series converges when $a < 0$ and diverges when $a \geq 0$.

5. For what values of a does the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+a}\right)^{n^2}$ converge? Here we write

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+a}\right)^{n^2} = \sum_{n=1}^{\infty}\frac{1}{(1+\frac{a}{n})^{n^2}}.$$

As in the previous case, we apply the root test and find

$$\lim_{n\to\infty}\left(\frac{1}{(1+\frac{a}{n})^{n^2}}\right)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{(1+\frac{a}{n})^n} = \frac{1}{e^a}.$$

So we conclude that $\sum_{n=1}^{\infty}\left(\frac{n}{n+a}\right)^{n^2}$ converges when $a > 0$.

6. $\sum_{n=1}^{\infty}(\sin\frac{1}{n})^n$

Here $a_n = (\sin\frac{1}{n})^n$ and hence $|a_n|^{\frac{1}{n}} = \sin(\frac{1}{n})$. Therefore the series $\sum_{n=1}^{\infty}(\sin\frac{1}{n})^n$ converges because $\lim_{n\to\infty}\sin(1/n) = 0 < 1$.

Review exercises from the textbook: Page 743: Problems 7 to 20 and 25 to 34. Page 746. Problems 1 to 38.

3.5. **Power Series:** Series of the form $\sum_{n=1}^{\infty} C_n(x-a)^n$ are called power series. We can use the ration or root test to find the values of x for which a power series converges.

Examples:

7. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n}$ converge?

We can use either the ratio or root test. Let's use the root test $a_n = \dfrac{(x-1)^n}{n^2 2^n}$ and so $|a_n|^{\frac{1}{n}} = \dfrac{|x-1|}{n^{\frac{2}{n}} 2}$. Since $n^{\frac{2}{n}} = (n^{\frac{1}{n}})^2$ and $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, it follows that $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \dfrac{|x-1|}{2}$.

The convergence is guaranteed if $|x-1| < 2$. This is an interval centered at 1 with radius 2, and this is called the radius of convergence. This interval can also be described as $-2 < x-1 < 2$ or $-1 < x < 3$. The result also says the series diverges for $|x-1| > 2$, or in other words if either $x > 3$ or $x < -1$. But what about the points $x = -1$ or $x = 3$? In this case the root test is inconclusive because the limit is equal to one. These cases have to be checked separately: When $x = -1$,

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \text{ which converges}$$

When $x = 3$,

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n} \sum_{n=1}^{\infty} \frac{(2)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ which converges}$$

Conclusion: The series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^2 2^n}$ converges if $-1 \leq x \leq 3$, or if x is on the interval $[-1, 3]$, and diverges if either $x > 3$ or $x < -1$. The interval $[-1, 3]$ is called the interval of convergence.

8. For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n 3^n}$ converge?

We will use the ratio test, but we could also use the root test. Notice that the series converges for $x = 3$ because all terms are equal to zero. So we may assume $x \neq 3$. In this case $a_n = \dfrac{(x-3)^n}{n 3^n}$ and therefore

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-3|^{n+1}}{(n+1)3^{n+1}} \frac{n 3^n}{|x-3|^n} = \frac{|x-3|}{3} \frac{n}{n+1}.$$

$$\lim_{n\to\infty} \left|\frac{a_n}{a_n + 1}\right| = \frac{|x-3|}{3}.$$

So the ratio test says that the series converges for $|x-3| < 3$ and diverges for $|x-3| > 3$. The radius of convergence is 3. Therefore the series converges if $-3 < x-3 < 3$ or $0 < x < 6$ and diverges if either $x > 6$ or $x < 0$. We need to check points $x = 0$ and $x - 6$ separately, because in these cases $\lim_{n\to 0} \left|\dfrac{a_{n+1}}{a_n}\right| = 1$ and the ratio test is inconclusive. So we need to test these points separately.

$$\text{when } x = 0, \quad \sum_{n=1}^{\infty} \frac{(x-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ which converges}$$

$$\text{when } x = 3, \quad \sum_{n=1}^{\infty} \frac{(x-3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{n3^n} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ which diverges .}$$

Conclusion: The series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n3^n}$ converges when $0 \leq x < 6$, and the interval of convergence is $[0, 6)$.

9. For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n n!}$ converge?

Here $a_n = \frac{(x-2)^n}{3^n n!}$. So for $x \neq 2$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|(x-2)|^{n+1}}{3^{n+1}(n+1)!} \frac{3^n n!}{|x-2|^n} = \frac{|x-2|}{3(n+1)}.$$

So we conclude that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 0 < 1$ for any x. Since the limit is always equal to zero, the series converges for every x.

Review exercises from the textbook: Page 751. Problems 1 to 26.

3.6. **Representation of functions as power series:** We say that a function $f(x)$ has a power series representation centered at a which has a radius of convergence R if the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges if $|x-a| < R$ and

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ for all } x \text{ satisfying } |x-a| < R.$$

Main result: If a function $f(x)$ has a power series representation centered at 0,

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \text{ if } |x| < R,$$

this is said to be **the** Maclaurin series of f and $c_n = \dfrac{f^{(n)}(0)}{n!}$. On the other hand, if a function $f(x)$ has a power series representation centered at a,

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n \text{ if } |x-a| < R,$$

this is said to be **the** Taylor series of f centered at a, or the Taylor series of f at a, and $b_n = \dfrac{f^{(n)}(a)}{n!}$.

Notice that this shows that a function $f(x)$ cannot have two distinct power series representations centered at the same point.

Here we will use the sum of the geometric series to construct many examples of functions which are represented by power series. Recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ provided } |x| < 1, \tag{3.1}$$

which is the Maclaurin series of $\dfrac{1}{1-x}$. We will use this to construct several other examples of converging power series. **Examples:**

1. Find the Maclaurin series representation of $\dfrac{1}{1+x}$. If we substitute x by $-x$ in the formula (3.1) above, we obtain

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n \text{ provided } |x| < 1.$$

2. Find the Maclaurin series representation of $\dfrac{1}{1+x^3}$. If we substitute x by $-x^3$ in (3.1), we obtain

$$\frac{1}{1+x^3} = \sum_{n=0}^{\infty}(-x^3)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n} \text{ provided } |x| < 1.$$

3. Find the Taylor series representation of $\dfrac{1}{x}$ centered at 2. Here the power series is centered at 2, which means we want an expression of the form

$$\frac{1}{x} = \sum_{n=0}^{\infty} c_n(x-2)^n.$$

The idea is to write

$$\frac{1}{x} = \frac{1}{2+x-2} = \frac{1}{2}\left(\frac{1}{1+\frac{x-2}{2}}\right)$$

If we use (3.1) with x replaced by $-\frac{x-2}{2}$, we have

$$\frac{1}{1+\frac{x-2}{2}} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{x-2}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}(x-2)^n, \text{ provided } \frac{|x-2|}{2} < 1.$$

Conclusion: $\dfrac{1}{x} = \dfrac{1}{2}\left(\dfrac{1}{1+\frac{x-2}{2}}\right) = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-2)^n,$ provided $\dfrac{|x-2|}{2} < 1.$

4. Find the Maclaurin series representation of $\dfrac{1}{x^2+5x+6}$. We begin by expanding $\dfrac{1}{x^2+5x+6}$ into partial fractions $\dfrac{1}{x^2+5x+6} = \dfrac{1}{x+2} - \dfrac{1}{x+3}$ and then write

$$\frac{1}{x^2+5x+6} = \frac{1}{x+2} - \frac{1}{x+3} = \frac{1}{2}(\frac{1}{1+\frac{x}{2}}) + \frac{1}{3}(\frac{1}{1+\frac{x}{3}}). \text{ But,}$$

$$\frac{1}{2}(\frac{1}{1+\frac{x}{2}}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n, \quad \text{provided } |x| < 2,$$

$$\frac{1}{3}(\frac{1}{1+\frac{x}{3}}) = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n, \quad \text{provided } |x| < 3.$$

Then in the smallest of the intervals $|x| < 2$ and $|x| < 3$, which is $|x| < 2$, both representations hold and we can write

$$\frac{1}{x^2+5x+6} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n, \text{ provided } |x| < 2.$$

Conclusion: $\dfrac{1}{x^2+5x+6} = \displaystyle\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2^{n+1}} + \frac{(-1)^n}{3^{n+1}}\right)x^n$ as long as $|x| < 2$.

5. Find the Taylor series representation of $f(x) = \dfrac{1}{x^2+6x+13}$ centered at -3. Just notice that

$$\frac{1}{x^2+6x+13} = \frac{1}{(x+3)^2+4} = \frac{1}{4}(\frac{1}{1+\frac{(x+3)^2}{4}}) = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{(x+3)^2}{4}\right)^n,$$

$$\text{provided } \frac{(x+3)^2}{4} < 1.$$

Conclusion: $\dfrac{1}{x^2+6x+13} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x+3)^{2n}$, provided $|x+3| < 2$.

The following result gives us a way of constructing even more examples of functions that are represented by power series.

Result: If $f(x)$ has a power series representation

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ provided } |x-a| < R,$$

Then the derivative and integral of f also have power series representations given by

$$f'(x) = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}, \text{ provided } |x-a| < R,$$

$$\int f(x)\, dx = C + \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}, \text{ provided } |x-a| < R.$$

Examples:

1. Find the Maclaurin series for $\frac{1}{(1-x)^3}$. We have to fist notice that $\frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$ and $\frac{d^2}{dx^2}\frac{1}{1-x} = \frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3}$. On the other hand,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1,$$

$$\frac{d}{dx}\frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1,$$

$$\frac{d^2}{dx^2}\frac{1}{1-x} = \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \quad |x| < 1.$$

Therefore, $\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\frac{1}{1-x} = \sum_{n=2}^{\infty} \frac{n(n-1)}{2}x^{n-2}, \quad |x| < 1$. One may want to express this in terms of x^n instead of x^{n-2}. Just set $k = n-2$ and then $n = k+2$ and so $\frac{1}{(1-x)^3} = \sum_{k=0}^{\infty} \frac{(k+2)(k+1)}{2}x^k$, provided $|x| < 1$.

2. Find the Maclaurin series expansion of $\ln(1-x)$. We know that

$$\ln(1-x) = C - \int \frac{1}{1-x}\, dx,$$

But from the result just stated

$$\int \frac{1}{1-x}\, dx = C + \int (\sum_{n=0}^{\infty} x^n)\, dx = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x| < 1$$

Therefore

$$\ln(1-x) = C + \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}, \quad |x| < 1.$$

If we set $x = 0$, we find that $C = \ln 1 = 0$.

Conclusion: $\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$, provided $|x| < 1$. If we set $x = \frac{1}{2}$ in this formula, we obtain $\ln(\frac{1}{2}) = -\ln 2 = -\sum_{n=0}^{\infty} \frac{\frac{1}{2}^{n+1}}{n+1}$ and so

$$\ln 2 = \sum_{n=0}^{\infty} \frac{1}{(n+1)2^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n2^n}. \text{ Isn't this cool?}$$

3. Find the Taylor series expansion of $\ln x$ centered at 10. We start from the fact that $\ln x = C + \int \frac{1}{x}\, dx$ and

$$\frac{1}{x} = \frac{1}{10 + x - 10} = \frac{1}{10}\left(\frac{1}{1 + \frac{x-10}{10}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}}(x-10)^n, \ |x - 10| < 10.$$

Therefore, provided $|x - 10| < 10$,

$$\ln x = C + \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{10^{n+1}}(x-10)^n\right) dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)10^{n+1}}(x-10)^{n+1}$$

To compute C, we just set $x = 10$ in this formula, so $C = \ln 10$.

Conclusion: $\ln x = \ln 10 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)10^{n+1}}(x-10)^{n+1}$, provided $|x - 10| < 10$.

4. Find the Maclaurin series representation of $\arctan x$. We use that

$$\arctan x = C + \int \frac{1}{1+x^2}\, dx = C + \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right) dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad |x| < 1.$$

If we set $x = 0$ we find that $C = 0$. So $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$, provided $|x| < 1$.

5. Compute $\arctan 0.1$ with an error not greater than 10^{-6}. We use the formula we just obtained and substitute $x = 0.1$. We obtain

$$\arctan 0.1 = \sum_{n=0}^{\infty} (-1)^n \frac{(0.1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{10^{2n+1}(2n+1)} =$$

Notice that this is an alternating series and we recall that if S_N is the sum of the first N terms of an alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$, then $|S - S_N| \leq b_{N+1}$. To apply this result consistently, we should rewrite the series so the sum starts at $n = 1$ If we set $k = n + 1$, when $n = 0$, then $k = 1$. But then $n = k - 1$, and so

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{10^{2n+1}(2n+1)} = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{10^{2k-1}(2k-1)}.$$

In this case, $b_k = \frac{1}{(2k-1)10^{2k-1}}$ and so we want

$$b_{N+1} = \frac{1}{(2N+1)10^{2N+1}} \leq \frac{1}{10^6},$$

which implies that $(2N + 2)10^{2N+2} \geq 10^6$. $N = 2$ does not quite do this, but $N = 3$ certainly does. So

Conclusion: $\arctan 0.1 = 0.1 - \frac{1}{3\ 10^3} + \frac{1}{5\ 10^5} = \frac{1}{10} - \frac{1}{3000} + \frac{1}{500000} + e$, where $|e| \leq 10^{-6}$.

Review exercises from the textbook: Page 757: Problems 3 to 9, 11, 12, 15, 16, 17, 18 19, 20, 21, 23 24, 29, 30.

3.7. **More Taylor and Maclaurin series:** The examples above are examples of Taylor and Maclaurin expansions. There are other functions which have Taylor and Maclaurin expansions:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad \text{for all } x \text{ on } (-\infty, \infty),$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \quad \text{for all } x \text{ on } (-\infty, \infty),$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n, \quad \text{for all } x \text{ on } (-\infty, \infty),$$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \frac{k(k-1)(k-2)(k-3)}{4!}x^4 + \dots$$
$$\frac{k(k-1)(k-2)(k-3)\dots(k-n+1}{n!}x^n + \dots |x| < 1.$$

We have adopted the convention: $0! = 1$ and we just use this convention to be able to write the Maclaurin series of e^x and $\cos x$ as above. With these formulas we can find the Taylor and Maclaurin series of variations of these functions.

Find the Maclaurin series of the following functions:

1. $\sin(3x)$ We just replace x with $3x$ in the Maclaurin series of $\sin x$. So we obtain

$$\sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(3x)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+1}, \quad \text{for all x}$$

2. $\cos(\dfrac{x^2}{4})$. Here we just replace x with $\frac{x^2}{4}$ in the Maclaurin series of $\cos x$. So we obtain

$$\cos(\frac{x^2}{4}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\frac{x^2}{4})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{2n}(2n)!}x^{4n}, \text{ for all x}$$

3. $e^{4x} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(4x)^n = \sum_{n=0}^{\infty} \frac{4^n}{n!}x^n$, for all x.
4. $e^{2x} + e^{3x}$. We write the Maclaurin series for each function separately

$$e^{2x} = \sum_{n=0}^{\infty} \frac{1}{n!}(2x)^n = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n,$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n, \text{ therefore}$$

$$e^{2x} + e^{3x} = \sum_{n=0}^{\infty} \frac{2^n}{n!}x^n + \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{2^n + 3^n}{n!}x^n.$$

5. $\dfrac{e^x - 1}{x}$. We know that $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}x^n$, for all x. Therefore $e^x - 1 = x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}x^n$ and therefore, $\dfrac{e^x - 1}{x} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}x^{n-1}$. If we set $k = n - 1$, and so $n = k + 1$, we obtain

$$\frac{e^x - 1}{x} = \sum_{k=0}^{\infty} \frac{1}{(k+1)!}x^k, \quad \text{for all } x.$$

6. Let $f(x) = \ln(x-1)$. Find $f^{(10)}(3)$ (the tenth derivative of f at 3). One could compute ten derivatives of the function and evaluate it at 3, but this is a lot of work. One can use the power series representation of this function centered at 3 and use its coefficients to compute the derivative of the function $f(x)$ at 3. We write

$$\frac{1}{x-1} = \frac{1}{(x-3)+2} = \frac{1}{2}(\frac{1}{1 + \frac{x-3}{2}}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-3)^n, \quad \frac{|x-3|}{2} < 1.$$

Therefore

$$\ln(x-1) = C + \int \frac{1}{x-1}dx = C + \int (\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-3)^n)\, dx =$$

$$C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}}(x-3)^{n+1}.$$

Setting $x = 3$ we obtain $C = \ln 2$. So we conclude that

$$\ln(x-1) = \ln 2 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^{n+1}}(x-3)^{n+1}.$$

On the other hand the Taylor series is defined to be of the form $\sum_{n=0}^{\infty} C_n(x-3)^n$, where $C_n = \dfrac{f^{(n)}(3)}{n!}$. So to find the tenth derivative, we have to look for the coefficient of $(x-3)^{10}$, which in this case is $C_{10} = \dfrac{(-1)^9}{10\, 2^{10}} = \dfrac{f^{(10)}(3)}{10!}$. Therefore

$$f^{(10)}(3) = -\frac{10!}{10\, 2^{10}}.$$

7. Represent $\int_0^1 \cos x^3\, dx$ as an infinite series. We know that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and therefore $\cos x^3 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n}$ and therefore

$$\int_0^1 \cos x^3\, dx = \int_0^1 (\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n})\, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\int_0^1 x^{6n}\, dx) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(6n+1)(2n)!}.$$

8. Find the Maclaurin series of $(1+x^2)^{\frac{1}{2}}$. We just have to use the formula above for $k = \frac{1}{2}$ and with x replaced by x^2.

$$(1+x^2)^{\frac{1}{2}} = 1 + hax^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + \dots$$
$$\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\dots(\frac{1}{2}-n+1}{n!}x^{2n} + \dots |x| < 1.$$

One may want to simplify these coefficients. Just notice that the coefficient of x^{2n} is equal to

$$\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\dots(\frac{1}{2}-n+1)}{n!} = \frac{1}{n!}\left(\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n+1)\right) =$$
$$\frac{1}{n!}\left(\frac{1}{2}(\frac{1-2}{2})(\frac{1-4}{2})\dots(\frac{1-2n+2}{2})\right) = \frac{1}{n!}\frac{1}{2^n}(1(-3)(-5)(-7)\dots(3-2n) =$$
$$\frac{(-1)^{n-1}}{n!2^n}1.3.5.\dots(2n-3).$$

Conclusion:

$$(1+x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n!2^n}1.3.5.\dots(2n-3)x^{2n}.$$

Review exercises from the textbook: Page 771. Problems number 3, 4, 5, 6, 8, 9, 11, 12, 16, 18, 21, 23, 31, 32, 35, 36, 37, 39, 40, 41.

4. Additional Exercises (mostly from old exams)

1. If $\{a_n\}$ is a sequence such that $\lim_{n\to\infty} n^3|a_n| = 1$. What can be said about the absolute convergence of $\sum_{n=1}^{\infty} a_n$?
2. Does $\sum_{n=1}^{\infty} n\sin(\frac{1}{n^3})$ converge or diverge?
3. Let $f(x)$ be a function defined on $[1,\infty)$ be such that $f(x) > 1$ for all x and that $\lim_{x\to\infty} \frac{f(x)}{x^2} = 1$. For what values of p does the series $\sum_{n=1}^{\infty} \sin(\frac{1}{f(n^p)})$?
4. Does $\sum_{n=1}^{\infty} n\sin(\frac{1}{n^3})$ converge or diverge?
5. Let $f(x) > 1$ for all x on $[1,\infty)$ and suppose that $\lim_{x\to\infty} \frac{f(x)}{x} = 10$. What can be said about the convergence of $\sum_{n=1}^{\infty} \tan(\frac{1}{f(n)})$ and $\sum_{n=1}^{\infty} \tan((\frac{1}{(f(n)})^3)$?
6. Let $f(x)$ be a function such that $f'(x) = x^2\cos x^2$ and $f(0) = 0$. Find the Maclaurin series of $f(x)$.
7. Express the integral $\int_0^1 \frac{e^x - 1}{x}\, dx$ as an infinite series.
8. If $f(x) = (1-x^2)^{-1}$ then $f^{(10)}(0) =?$
9. Find the values of p for which the series $\sum_{n=1}^{\infty} \sqrt{\frac{3}{1+n^p}}$ converges?
10. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{2^n x^n}{n+1}$.
11. Express $\int_0^x \frac{t}{1-t^3}\, dt$ as a power series in x and find its radius of convergence
12. Let $\{a_n\}$ be a sequence such that $a_1 > 0$ and $a_{n+1} = (-1)^n \frac{n+1}{2n+7}$ for $n \geq 1$. Let $\{b_n\}$ be a sequence with $b_n > 0$ such that $\lim_{n\to\infty} \frac{|a_n|}{b_n} = 3$ what can be said about the convergence of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$?
13. What can be said about the convergence of $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{9+3n}{7n}\right)^n$?
14. What can be said about the convergence of $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{9+3n}{7n}\right)^{n^2}$?

15. Find the Taylor series representation of $f(x) = \dfrac{x-2}{x^2-4x+5}$ centered at 2 and its radius of convergence.
16. Knowing that $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$, find the smallest number of terms that one needs to compute $\ln(1.1)$ with an error less than or equal to 10^{-8}?

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