MA 26100 - Spring 2017

Study Guide # 2

1. Relative/local extrema; critical points (points where $\nabla f = \vec{\mathbf{0}}$ or ∇f does not exist).

2. 2^{nd} Derivatives Test: Suppose the 2^{nd} partials of $f(x, y)$ are continuous in a disk with center (a, b) and $\nabla f(a, b) = \vec{\mathbf{0}}$. Let $D = \left| \begin{matrix} f_{xx} & f_{xy} \\ f_{yy} & f_{yx} \end{matrix} \right|_{(a,b)}$.

(a) If $D > 0$ and $f_{xx}(a, b) > 0 \implies f(a, b)$ is a local minimum value.

(b) If $D > 0$ and $f_{xx}(a, b) < 0 \implies f(a, b)$ is a local maximum value.

(c) If $D < 0 \implies f(a, b)$ is a *not* a local min or local max value. So (a, b) is a **saddle point** of f.

If $D = 0$ (or if $\nabla f(a, b)$ does not exist or f has more than 2 variables) the test gives no information and you need to do something else to determine if a relative extremum exists.

3. Absolute extrema; Max-Min Problems.

4. Double integrals; Midpoint Rule for rectangle : $\displaystyle\iint_R f(x, y)\, dA \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(\overline{x_i}, \overline{y_j})\ \Delta A$;

5. Type I region $D : \left\{ \begin{matrix} g_1(x) \le y \le g_2(x) \\ a \le x \le b \end{matrix} \right.$; Type II region $D : \left\{ \begin{matrix} h_1(y) \le x \le h_2(y) \\ c \le y \le d \end{matrix} \right.$;

iterated integrals over Type I and II regions: $\displaystyle\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$ and $\displaystyle\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$, respectively; Reversing Order of Integration (regions that are both Type I and Type II); properties of double integrals.

6. Integral inequalities: $mA \le \displaystyle\iint_D f(x, y)\, dA \le MA$, where $A =$ area of D and $m \le f(x, y) \le M$ on D.

7. Change of Variables Formula in Polar Coordinates: if $D : \left\{ \begin{matrix} h_1(\theta) \le r \le h_2(\theta) \\ \alpha \le \theta \le \beta \end{matrix} \right.$, then

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\, \underset{\uparrow}{r}\, dr\, d\theta.$$

8. Applications of double integrals:

(a) Area of region D is $A(D) = \iint_D dA$

(b) Volume of solid under graph of $z = f(x,y)$, where $f(x,y) \geq 0$, is $V = \iint_D f(x,y)\, dA$

(c) Mass of D is $m = \iint_D \rho(x,y)\, dA$, where $\rho(x,y) =$ density (per unit area); sometimes write $m = \iint_D dm$, where $dm = \rho(x,y)\, dA$.

(d) Moment about the x-axis $M_x = \iint_D y\,\rho(x,y)\, dA$; moment about the y-axis $M_y = \iint_D x\,\rho(x,y)\, dA$.

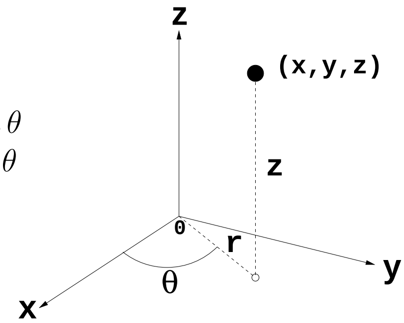
(e) Center of mass $(\overline{x}, \overline{y})$, where $\overline{x} = \dfrac{M_y}{m} = \dfrac{\iint_D x\,\rho(x,y)\, dA}{\iint_D \rho(x,y)\, dA}$, $\overline{y} = \dfrac{M_x}{m} = \dfrac{\iint_D y\,\rho(x,y)\, dA}{\iint_D \rho(x,y)\, dA}$

Remark: centroid = center of mass when density is constant (this is useful).

9. Elementary solids $E \subset \mathbb{R}^3$ of Type 1, Type 2, Type 3; triple integrals over solids E:
$\iiint_E f(x,y,z)\, dV = \iint_D \int_{u(x,y)}^{v(x,y)} f(x,y,z)\, dz\; dA$ for $E = \{(x,y) \in D,\; u(x,y) \leq z \leq v(x,y)\}$;
volume of solid E is $V(E) = \iiint_E dV$; applications of triple integrals, mass of a solid, moments about the coordinate planes M_{xy}, M_{xz}, M_{yz}, center of mass of a solid $(\overline{x}, \overline{y}, \overline{z})$.

10. Cylindrical Coordinates (r, θ, z):

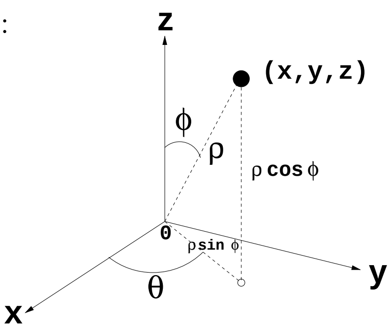
From CC to RC : $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$



Going from RC to CC use $x^2 + y^2 = r^2$ and $\tan\theta = \dfrac{y}{x}$ (make sure θ is in correct quadrant).

11. Spherical Coordinates (ρ, θ, ϕ), where $0 \leq \phi \leq \pi$:

From SC to RC : $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$



Going from RC to SC use $x^2 + y^2 + z^2 = \rho^2$, $\tan\theta = \dfrac{y}{x}$ and $\cos\phi = \dfrac{z}{\rho}$.

12. Triple integrals in <u>Cylindrical Coordinates</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $\boldsymbol{dV = r\,dz\,dr\,d\theta}$

$$\iiint_E f(x,y,z)\ dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$
$$\uparrow$$

13. Triple integrals in <u>Spherical Coordinates</u>: $\begin{cases} x = (\rho\sin\phi)\cos\theta \\ y = (\rho\sin\phi)\sin\theta \\ z = \rho\cos\phi \end{cases}$, $\boldsymbol{dV = \rho^2 \sin\phi \;\; d\rho\, d\phi\, d\theta}$

$$\iiint_E f(x,y,z)\ dV = \iiint_E f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$
$$\uparrow$$

14. Vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$: $\vec{\mathbf{F}}(x,y) = \langle P(x,y),\ Q(x,y)\rangle$ and
$\vec{\mathbf{F}}(x,y,z) = \langle P(x,y,z),\ Q(x,y,z),\ R(x,y,z)\rangle$;
$\vec{\mathbf{F}}$ is a conservative vector field if $\vec{\mathbf{F}} = \nabla f$, for some real-valued function f.

15. Line integral of a function $f(x,y)$ along C, parameterized by $x = x(t)$, $\;y = y(t)$ and $a \le t \le b$, is

$$\int_C f(x,y)\ ds = \int_a^b f(x(t),\, y(t))\ \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt\,.$$

(independent of orientation of C, other properties and applications of line integrals of f)

<u>**Remarks**</u>:

(a) $\displaystyle\int_C f(x,y)\, ds$ is sometimes called the *"line integral of f with respect to arc length"*

(b) $\displaystyle\int_C f(x,y)\, dx = \int_a^b f(x(t),y(t))\, x'(t)\, dt$

(c) $\displaystyle\int_C f(x,y)\, dy = \int_a^b f(x(t),y(t))\, y'(t)\, dt$

13. Line integral of vector field $\vec{\mathbf{F}}(x,y)$ along C, parameterized by $\vec{\mathbf{r}}(t)$ and $a \le t \le b$, is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_a^b \vec{\mathbf{F}}(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t)\, dt\,.$$

(depends on orientation of C, other properties and applications of line integrals of f)

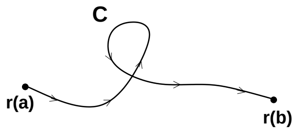
14. Connection between line integral of vector fields and line integral of functions:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C (\vec{\mathbf{F}} \cdot \vec{\mathbf{T}})\, ds$$

where $\vec{\mathbf{T}}$ is the unit tangent vector to the curve C.

15. If $\vec{\mathbf{F}}(x,y) = P(x,y)\,\vec{\mathbf{i}} + Q(x,y)\,\vec{\mathbf{j}}$, then $\displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_C P(x,y)\, dx + Q(x,y)\, dy$; Work $= \displaystyle\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.

16. <u>FUNDAMENTAL THEOREM OF CALCULUS FOR LINE INTEGRALS</u>: $\int_C \nabla f \cdot d\vec{\mathbf{r}} = f(\vec{\mathbf{r}}(b)) - f(\vec{\mathbf{r}}(a))$:



17. A vector field $\vec{\mathbf{F}}(x, y) = P(x, y)\,\vec{\mathbf{i}} + Q(x, y)\,\vec{\mathbf{j}}$ is *conservative* (i.e. $\vec{\mathbf{F}} = \nabla f$) if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$; how to determine a potential function f if $\vec{\mathbf{F}}(\vec{\mathbf{x}}) = \nabla f(\vec{\mathbf{x}})$.