Fall 2018

MA 16200
Study Guide - Exam # 2

(1) Techniques of Integration

(a) Trig Integrals: Integrals of the type $\int \sin^m x \cos^n x\,dx$ and $\int \tan^m x \sec^n x\,dx$

Some useful trig identities:

(i) $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$

(ii) $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$ and $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

(iii) $\sin 2\theta = 2\sin\theta\,\cos\theta$

Some useful trig integrals:

(i) $\int \tan u\ du = \ln|\sec u| + C$

(ii) $\int \sec u\ du = \ln|\sec u + \tan u| + C$

(b) Trig integrals of the form: $\int \sin mx\ \sin nx\ dx$, $\int \cos mx\ \cos nx\ dx$, $\int \sin mx\ \cos nx\ dx$,
use these trig identities:

$$\sin A\ \sin B = \frac{1}{2}\{\cos(A-B) - \cos(A+B)\}$$

$$\cos A\ \cos B = \frac{1}{2}\{\cos(A-B) + \cos(A+B)\}$$

$$\sin A\ \cos B = \frac{1}{2}\{\sin(A-B) + \sin(A+B)\}$$

(c) Trigonometric Substitutions:

*Expression**	*Trig Substitution*	*Identity needed*
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

* Or powers of these expressions.

(2) Integration via Partial Fractions: Use for (proper) rational functions $\dfrac{R(x)}{Q(x)}$;

If $\deg R(x) \geq \deg Q(x)$, i.e. rational function is improper, then do polynomial division before using partial fractions: $P(x) = S(x)\,Q(x) + R(x)$ where $\deg R(x) < \deg Q(x)$.

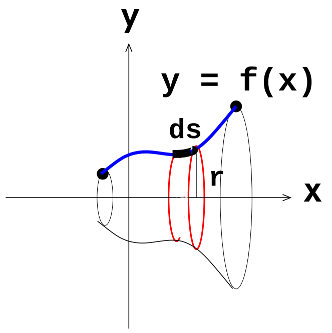
Shortcut for dividing by $Q(x) = x - c$: $S(x) = (P(x) - P(c))/(x-c)$, $R = P(c)$.

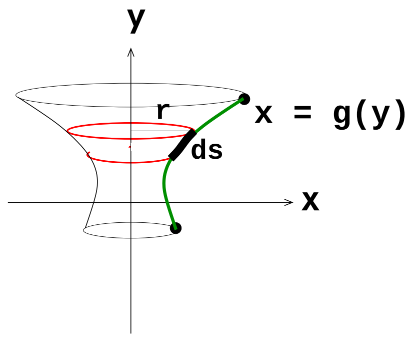
(3) Improper integrals: **Type I** (unbounded intervals) $\int_a^{\infty} f(x)\,dx$, $\int_{-\infty}^{b} f(x)\,dx$ or $\int_{-\infty}^{\infty} f(x)\,dx$;
Improper integrals of **Type II** (discontinuous integrand at one or both endpoints) $\int_a^b f(x)\,dx$.

Comparison Theorem: Let $f(x)$ and $g(x)$ be continuous for $x \geq a$.
(a) If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^{\infty} g(x)\,dx$ converges $\Longrightarrow \int_a^{\infty} f(x)\,dx$ also converges.
(b) If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and $\int_a^{\infty} g(x)\,dx$ diverges $\Longrightarrow \int_a^{\infty} f(x)\,dx$ also diverges.

(4) Arc length $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$ or $L = \int_c^d \sqrt{1 + (g'(y))^2}\,dy$.

(5) Surface area of revolution: $S = \int 2\pi\ \{\texttt{ribbon radius}\}\ ds$ or $S = \int 2\pi r\,ds$,
where $ds = \sqrt{1 + (f'(x))^2}\,dx$ or $ds = \sqrt{1 + (g'(y))^2}\,dy$.





(6) Center of mass of a system of discrete masses $m_1, m_2, \cdots, m_n$ located at $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ is $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}, \qquad \overline{y} = \frac{M_x}{M} = \frac{\sum_{k=1}^{n} m_k y_k}{\sum_{k=1}^{n} m_k}$$

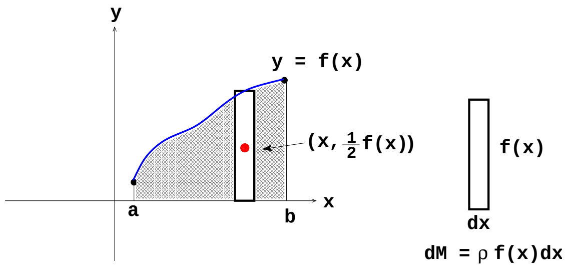
$M_x =$ moment of system about the $x-$axis; $M_y =$ moment of system about the $y-$axis;
$M =$ total mass of the system.

(7) Moments, center of mass (center of mass = *centroid* if density $\rho =$ constant).

(a) Lamina defined by $y = f(x), \;\; a \le x \le b$ and $\rho =$ constant:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x\rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

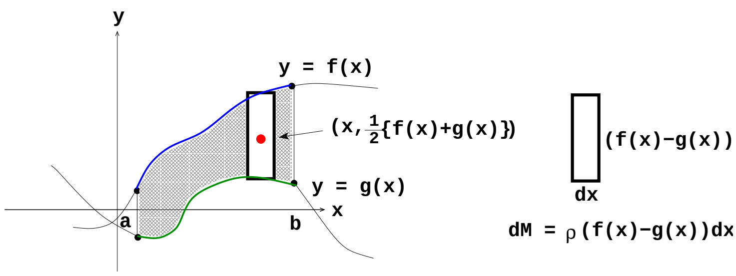
$$\overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\rho\left\{f(x)\right\}^2\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b \frac{1}{2}\left\{f(x)\right\}^2\,dx}{\int_a^b f(x)\,dx}$$



(b) Lamina between two curves by $y = f(x)$, $y = g(x), \;\; a \le x \le b$ and $\rho =$ constant:

$$\overline{x} = \frac{M_y}{M} = \frac{\int_a^b x\rho(f(x) - g(x))\,dx}{\int_a^b \rho(f(x) - g(x))\,dx} = \frac{\int_a^b x(f(x) - g(x))\,dx}{\int_a^b (f(x) - g(x))\,dx}$$

$$\overline{y} = \frac{M_x}{M} = \frac{\int_a^b \frac{1}{2}\rho\left(\{f(x)\}^2 - \{g(x)\}^2\right)\,dx}{\int_a^b \rho(f(x) - g(x))\,dx} = \frac{\int_a^b \frac{1}{2}\left(\{f(x)\}^2 - \{g(x)\}^2\right)\,dx}{\int_a^b (f(x) - g(x))\,dx}$$



(8) Sequences; limits of sequences; Limit Laws for Sequences; monotone sequences (increasing and decreasing); bounded sequences; `Monotone Sequence Theorem`.

(9) Additional useful limit theorems:

(a) *Theorem*: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, then $\lim_{n\to\infty} a_n = L$.

(b) *Squeeze Theorem for Sequences*: If $a_n \le b_n \le c_n$ for all $n \ge N_0$ with $a_n \longrightarrow L$ and $c_n \longrightarrow L$, then $b_n \longrightarrow L$.

(c) *Theorem*: If $a_n \longrightarrow L$ and f is continuous at L, then $f(a_n) \longrightarrow f(L)$.

(10) Infinite series $\sum_{n=1}^{\infty} a_n$; n^{th} partial sum $s_n = \sum_{k=1}^{n} a_k$; the infinite series $\sum_{n=1}^{\infty} a_n$ **converges** to s if $s_n \to s$; the infinite series **diverges** if $\{s_n\}$ does not have a limit.

(11) `Divergence Test for Series`: If $\lim_{n\to\infty} a_n \neq 0$ or limit fails to exist $\Longrightarrow \sum_{n=1}^{\infty} a_n$ DIVERGES.

(12) Special Infinite Series:

(a) `Harmonic Series`: $\sum_{n=1}^{\infty} \frac{1}{n}$. This series DIVERGES.

(b) `Geometric Series`: $\sum_{n=1}^{\infty} ar^{n-1}$

(i) $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots = a(1 + r + r^2 + r^3 + \cdots) = \frac{a}{1-r}$, **if** $|r| < 1$.

(ii) $\sum_{n=1}^{\infty} ar^{n-1}$ will DIVERGE **if** $|r| \ge 1$.