# Answer Keys for
# Study Guide for Final Exam

1. 1.

(i) $f(t) = \sqrt{5-t} + \dfrac{1}{\sqrt{t^2-4}}$

Conditions

- $5 - t \geqq 0$ i.e. $5 \geqq t$
- $t^2 - 4 \geqq 0$ i.e. $t \leqq -2,\ 2 \leqq t$
- $\sqrt{t^2-4} \neq 0$ i.e. $t \neq \pm 2$

Domain

$$(-\infty, -2) \cup (2, 5]$$

(ii) $f(x) = \dfrac{1}{\ln(x^2-1)}$

Conditions

- $x^2 - 1 > 0$ i.e. $x < -1,\ 1 < x$
- $\ln(x^2-1) \neq 0$ i.e. $x^2 - 1 \neq 1$
  i.e. $x \neq \pm\sqrt{2}$

Domain $(-\infty, -\sqrt{2}) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (\sqrt{2}, \infty)$

(iii) $\qquad f(x) = \sqrt{e^{2x} - 2e^{x} + \frac{3}{4}}$ $\qquad$ (2)

Condições

$$e^{2x} - 2e^{x} + \frac{3}{4} \geqq 0$$

$$\shortparallel$$

$$(e^{x})^2 - 2e^{x} + \frac{3}{4}$$

$$\shortparallel$$

$$(e^{x} - 1)^2 - \frac{1}{4}$$

$$\shortparallel$$

$$\left(e^{x} - 1 + \frac{1}{2}\right)\left(e^{x} - 1 - \frac{1}{2}\right)$$

$$\shortparallel$$

$$\left(e^{x} - \frac{1}{2}\right)\left(e^{x} - \frac{3}{2}\right)$$

i.e. $\quad e^{x} \leqq \frac{1}{2} \quad , \quad \frac{3}{2} \leqq e^{x}$

i.e. $\quad x \leqq \ln\left(\frac{1}{2}\right) = -\ln 2$

or $\quad \ln\left(\frac{3}{2}\right) = \ln 3 - \ln 2 \leqq x$

Domain

$$(-\infty, -\ln 2] \cup [\ln 3 - \ln 2, \infty)$$

(iv) $f(x) = \sqrt{\dfrac{1}{2 - \ln(x-1)}}$

Conditions

- $x - 1 > 0$ i.e. $x > 1$
- $2 - \ln(x-1) \neq 0$

i.e. $2 \neq \ln(x-1)$

i.e. $x - 1 \neq e^2$.

i.e. $x \neq e^2 + 1$.

- $\dfrac{1}{2 - \ln(x-1)} \geq 0$.

i.e. $2 - \ln(x-1) > 0$

i.e. $2 > \ln(x-1)$

i.e. $e^2 > x - 1$

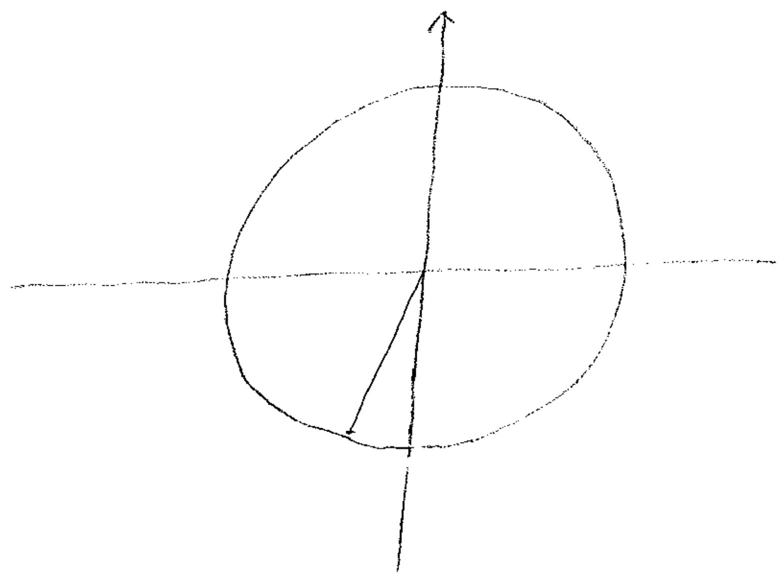
i.e. $e^2 + 1 > x$

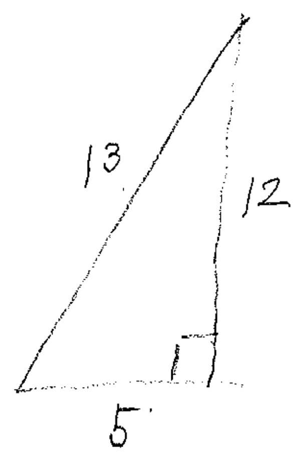
Domain

$(1, e^2+1)$

2.1.

$$\sin\theta = -\frac{12}{13}, \quad \pi < \theta < \frac{3\pi}{2}$$





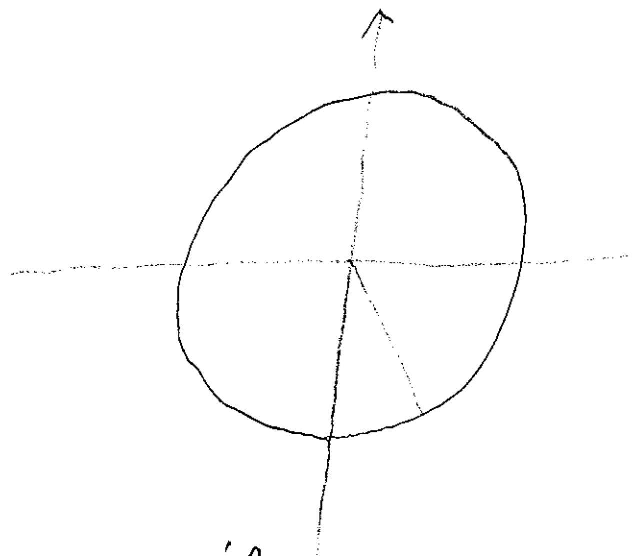

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$= \frac{-5/13}{-12/13}$$

$$= \frac{5}{12}$$

2.2

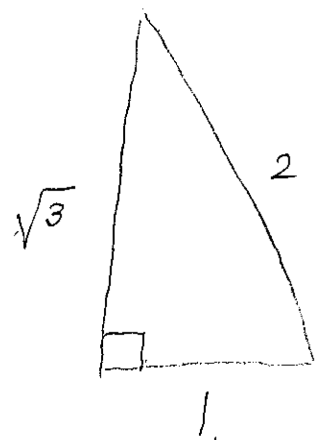
$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \frac{3\pi}{2} < \theta < 2\pi$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{-\sqrt{3}/2}{1/2}$$

$$= -\sqrt{3}$$

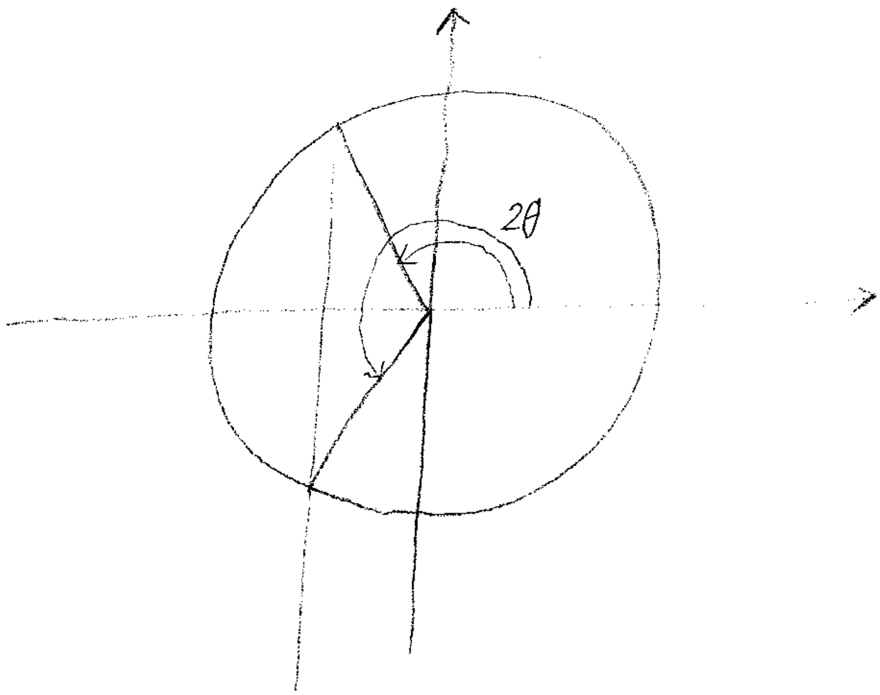


2 . 3 .

$$\cos 2\theta = -\frac{1}{2} \qquad 0 < \theta < \pi$$

i.e.

$$0 < 2\theta < 2\pi$$




$$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$$

i.e.

$$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$$

$$\sin\theta = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

or

$$\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

3.1.

$$\sqrt{3}\sin x = \sin(2x) \quad \text{on} \quad [0, 2\pi]$$

$$\sqrt{3}\sin x - \sin(2x) = 0$$

$$= \sqrt{3}\sin x - 2\sin x\cos x$$

$$= \sin x\left(\sqrt{3} - 2\cos x\right)$$

$$\begin{cases} \sin x = 0 \\ \text{or} \quad \cos x = \dfrac{\sqrt{3}}{2} \end{cases}$$

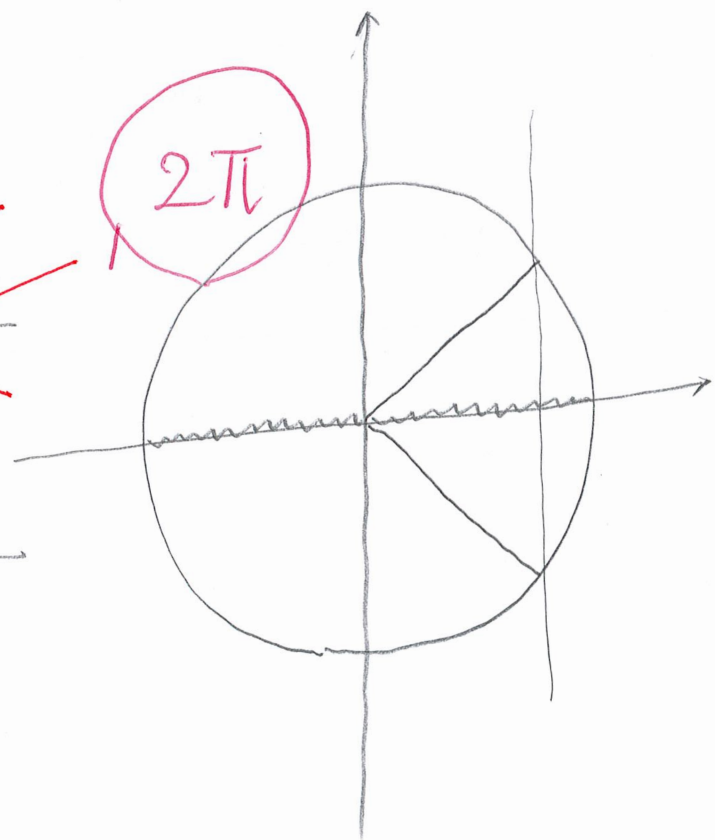
i.e.

$$x = \cancel{\frac{\pi}{2}}, \cancel{\frac{3\pi}{2}}$$

$$0, \pi, 2\pi$$

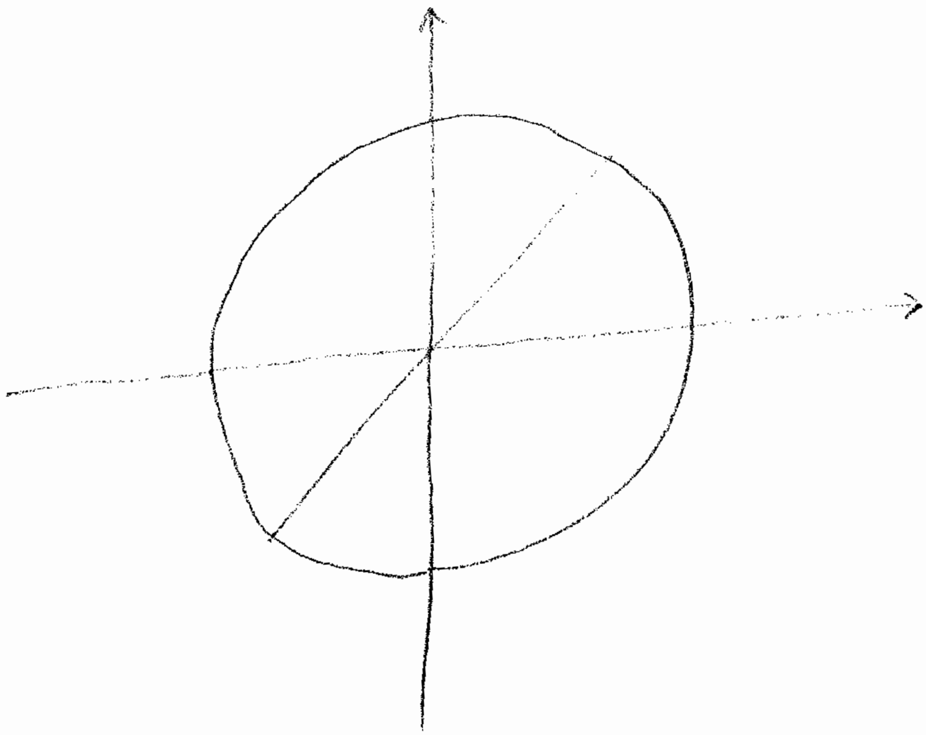
or

$$x = \frac{\pi}{6}, \frac{11\pi}{6}$$



3.2

$\cos(2x) - \sin(2x) = 0$ on $[0, 2\pi]$



$0 \leq x \leq 2\pi$

$0 \leq 2x \leq 4\pi.$

$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$x = \pi/8, \quad 5\pi/8, \quad 9\pi/8, \quad 13\pi/8$

4.1.

(i) $$f(x) = \frac{6x-1}{2x+1}$$

$$= 3 + \frac{-4}{2x+1}$$

Domain $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

Range $(-\infty, 3) \cup (3, \infty)$

Inverse function

Step 1 $$y = \frac{6x-1}{2x+1}$$

Step 2 $$(2x+1)y = 6x-1$$

$$(2y-6)x = -y-1$$

$$x = \frac{-y-1}{2y-6}$$

Step 3

$$f^{-1}(x) = \frac{-x-1}{2x-6}$$

Domain $(-\infty, 3) \cup (3, \infty)$

Range $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

(ii) $$f(x) = \frac{2e^x - 1}{2e^x + 1}.$$

Domain $(-\infty, \infty)$

In order to find the range, we'll see for what value(s) of $y$ the equation $y = \dfrac{2e^x - 1}{2e^x + 1}$ has a solution for $x$

$$y = \frac{2e^x - 1}{2e^x + 1}$$

$$(2e^x + 1)y = 2e^x - 1.$$

$$(2y - 2)e^x = -y - 1$$

$$e^x = \frac{-y-1}{2y-2}$$

$$x = \ln\left(\frac{-y-1}{2y-2}\right)$$

Condition

$$\frac{-y-1}{2y-2} > 0$$

We analyze this condition

$$\begin{cases} -y-1 > 0 & \text{when} \quad 2y-2 > 0. \\ \text{i.e.} & \text{i.e.} \\ -1 > y & y > 1. \end{cases}$$

$\longrightarrow$ no solution

$$\begin{cases} -y-1 < 0 & \text{when} \quad 2y-2 < 0 \\ \text{i.e.} & \text{i.e.} \\ -1 < y & y < 1. \end{cases}$$

$\longrightarrow \quad -1 < y < 1.$

Range $(-1, 1)$

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$$f^{-1}(x) = \ln\left(\frac{-x-1}{2x-2}\right)$$

Domain $(-1, 1)$

Range $(-\infty, \infty)$

(iii) $f(x) = 1 - \sqrt{x+1}$

Condition

$x + 1 \geqq 0$ i.e. $x \geqq -1$

Domain $[-1, \infty)$

Range $(-\infty, 1]$

Inverse function

Step 1. $y = 1 - \sqrt{x+1}$

Step 2. $y - 1 = -\sqrt{x+1}$

$1 - y = \sqrt{x+1}$

$(1-y)^2 = x + 1$

$x = (1-y)^2 - 1$

$= y^2 - 2y$

Step 3 $f^{-1}(x) = x^2 - 2x$

Domain $(-\infty, 1]$

Range $[-1, \infty)$

5.2

$$f(x) = \begin{cases} x^2 - a & x \le 1 \\ \dfrac{3x^2+12x-b}{x^2+2x-3} & x > 1. \end{cases}$$

The function is continuous at $x \neq 1$.

So the only issue is at $x = 1$

$f$ continuous at $x = 1$.

$\Longleftrightarrow$

$$\underbrace{\lim_{x\to 1^-} f(x) = f(1)}_{1^2 - a} = \lim_{x\to 1^+} f(x)$$

$$= \lim_{x\to 1^+} \frac{3x^2+12x-b}{x^2+2x-3}$$

Since $\lim_{x\to 1^+} (x^2+2x-3) = 0$,

for this limit to exist, we have to have

$$\underbrace{\lim_{x\to 1^+} (3x^2 + 12x - b)}_{\displaystyle = 3\cdot 1^2 + 12\cdot 1 - b} = 0$$

$$\longrightarrow \quad b = 15$$

Now we compute

$$\lim_{x\to 1^+} \frac{3x^2 + 12x - b}{x^2 + 2x - 3}$$

$$3(x^2 + 4x - 5)$$

$$= \lim_{x\to 1^+} \frac{3x^2 + 12x - 15}{x^2 + 2x - 3}$$

$$= \lim_{x\to 1^+} \frac{3(x-1)(x+5)}{(x-1)(x+3)} = \frac{9}{2}$$

$$\longrightarrow \quad 1 - a = \frac{9}{2}$$

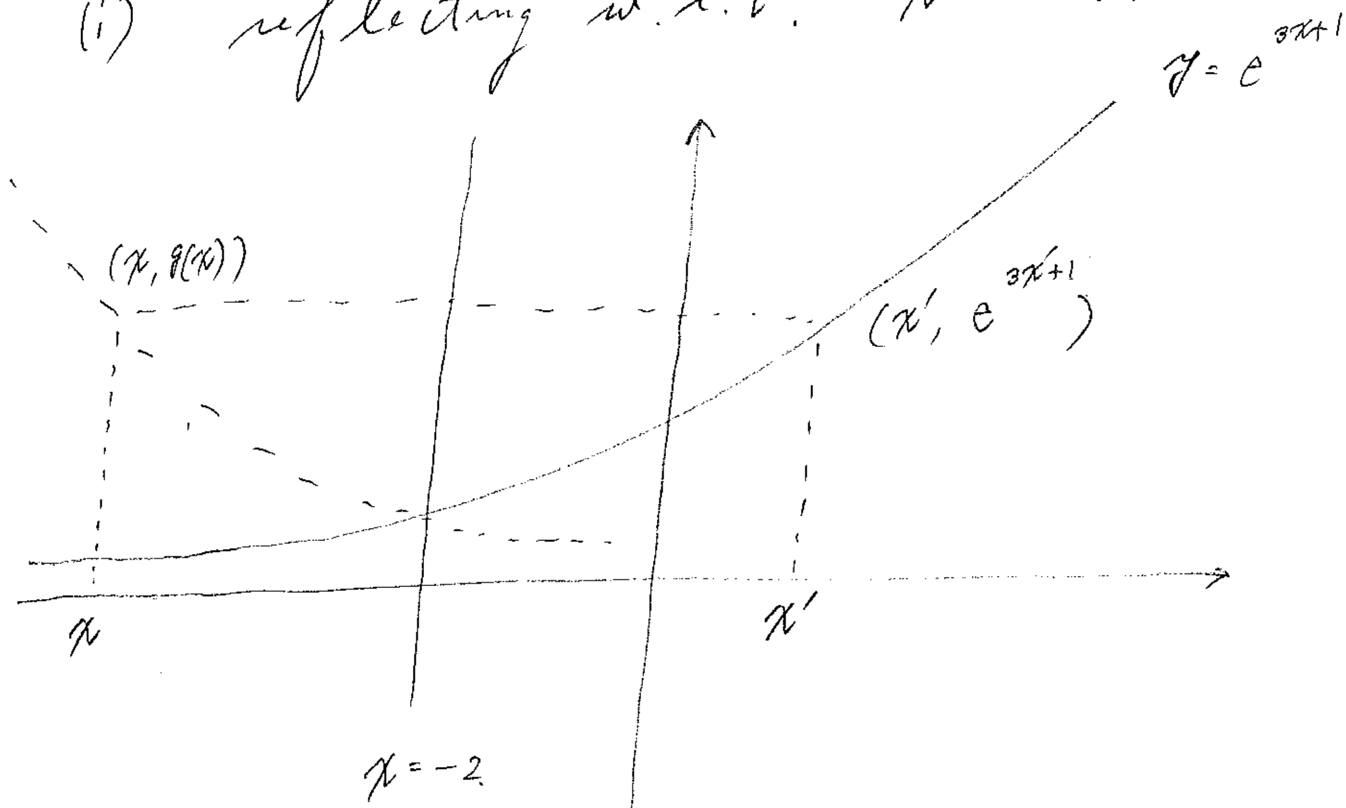
$$\longrightarrow \quad a = 1 - \frac{9}{2} = -\frac{7}{2}$$

Ans. $a = -\dfrac{7}{2}$, $b = 15$

6.1

$$y = e^{3x+1}$$

(i) reflecting w.r.t. $x = -2$




$$\frac{x + x'}{2} = -2$$

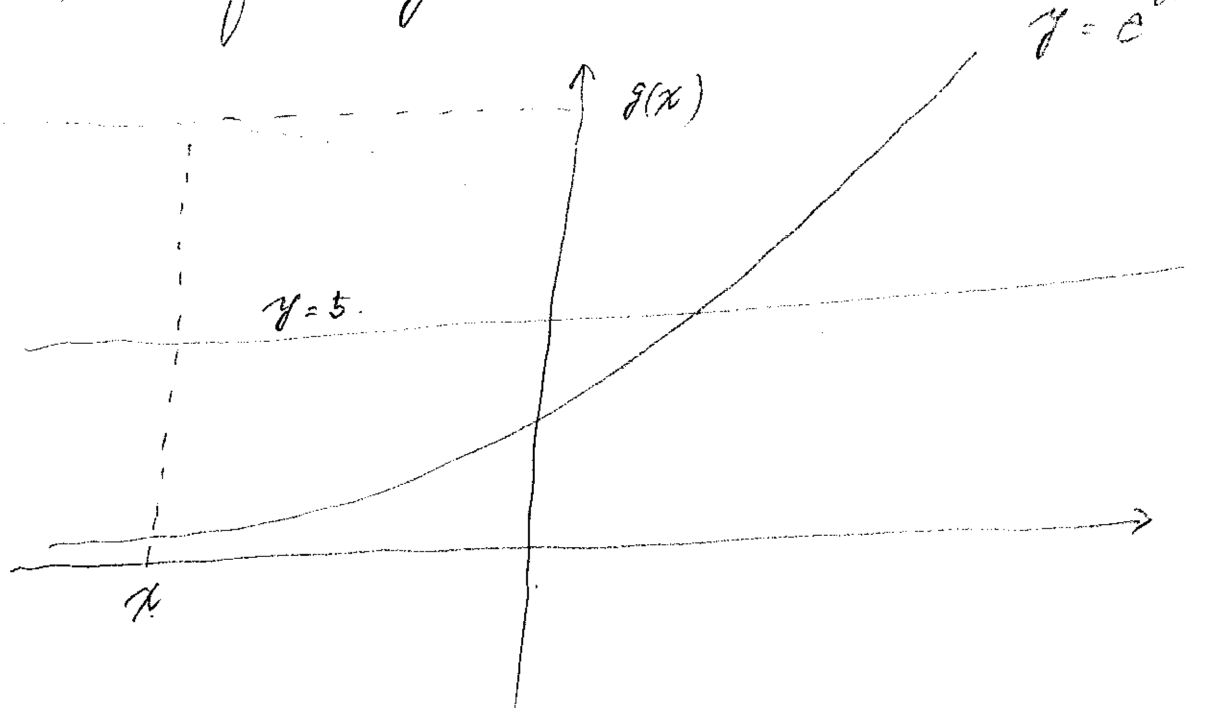
$$\longrightarrow x' = -x - 4$$

$$g(x) = e^{3x'+1}$$

$$= e^{3(-x-4)+1} = e^{-3x-11}$$

Ans. $y = e^{-3x-11}$

(ii) reflecting w.r.t. $y = 5$




$$\frac{g(x) + e^{3x+1}}{2} = 5$$

$$g(x) = 10 - e^{3x+1}$$

Ans. $y = 10 - e^{3x+1}$

(iii) reflecting w.r.t. $y = x$

The graph of $y = g(x)$ is the graph of the inverse function of $y = e^{3x+1}$

Step 1. $y = e^{3x+1}$

Step 2. $\ln y = 3x + 1$

$\ln y - 1 = 3x$

$\frac{\ln y - 1}{3} = x$

Step 3. $y = \frac{\ln x - 1}{3}$

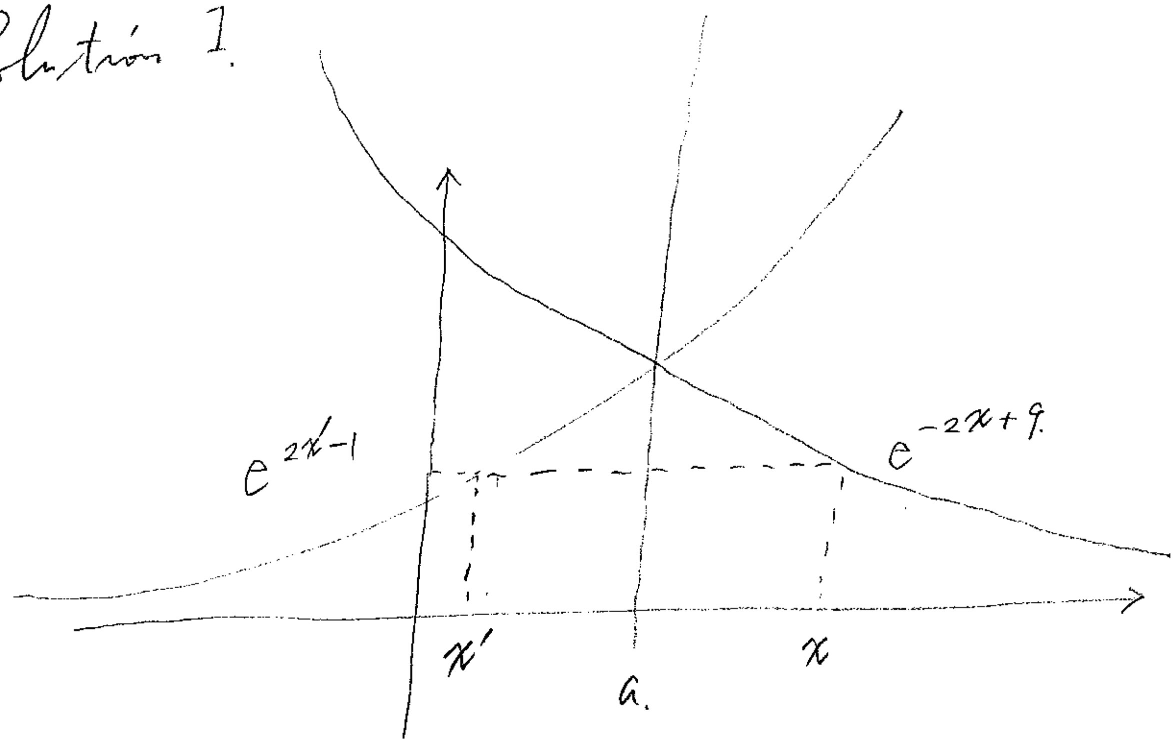
Ans. $y = \frac{\ln x - 1}{3}$

6.2. $y = e^{2x-1}$.

Reflect it w.r.t. $x = a$.

The resulting eq. is $y = e^{-2x+9}$.

Solution 1.




$$\begin{cases} \dfrac{x+x'}{2} = a \rightarrow x' = 2a - x \\ e^{-2x+9} = e^{2x'-1} \rightarrow e^{2x'-1} \end{cases}$$

$$= e^{2(2a-x)-1}$$

$$= e^{-2x+4a-1}$$

i.e.,

$$e^{-2x+9} = e^{-2x+4a-1}.$$

$$-2x + 9 = -2x + 4a - 1$$

$$\rightarrow \quad 9 = 4a - 1.$$

$$\rightarrow \quad a = \frac{5}{2}$$

Solution 2.

At $x = a$,

the value of $y = e^{2x-1}$

and : $y = e^{-2x+9}$.

must be equal.

$$\rightarrow \quad e^{2a-1} = e^{-2a+9}$$

$$\rightarrow \quad 2a - 1 = -2a + 9$$

$$\rightarrow \quad a = \frac{5}{2}.$$

7.1.

a function $f(x)$ with $f'(2) = 5$.

(i) $g(x) = f(2x) = f(h(x))$

where $h(x) = 2x$.

$$g'(1) = f'(h(1)) \cdot h'(1)$$
$$= f'(2 \cdot 1) \cdot 2 = 5 \cdot 2 = 10$$

(ii) $$\lim_{h \to 0} \frac{f(2+4h) - f(2)}{3h}$$

$$= \lim_{h \to 0} \underbrace{\frac{f(2+4h) - f(2)}{4h}}_{\to f'(2)} \cdot \frac{4h}{3h}$$

$$= f'(2) \cdot \frac{4}{3} = 5 \cdot \frac{4}{3} = \frac{20}{3}$$

(iii) $$\lim_{h \to 0} \frac{f(2+4h) - f(2+5h)}{9h}$$

$$= \lim_{h \to 0} \frac{f(2+4h) - f(2+5h)}{\underbrace{(2+4h) - (2+5h)}_{-h}} \cdot \frac{-h}{9h}$$

$$\to f'(2)$$

$$= f'(2) \cdot \left(-\frac{1}{9}\right) = 5 \cdot \left(-\frac{1}{9}\right) = -\frac{5}{9}$$

8. 1

$$y = f(x) = \frac{2}{3} x\sqrt{x}$$

$$= \frac{2}{3} x^{\frac{3}{2}}$$

$$f'(x) = \frac{2}{3} \cdot \frac{3}{2} x^{\frac{3}{2}-1}$$

$$= \sqrt{x}$$

parallel to $y = 2x + 3$

$\Leftrightarrow$ $f'(x) = \sqrt{x} = 2$

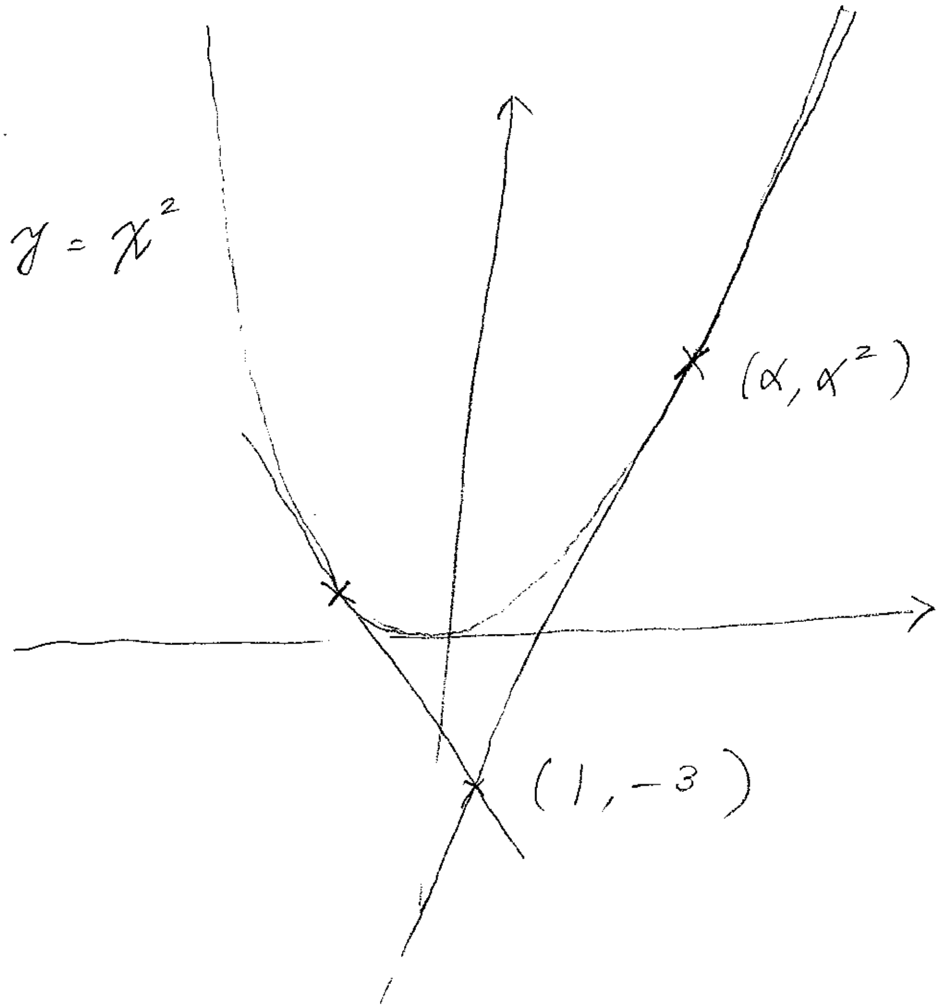
$\longrightarrow$ $x = 4$ $\left(\rightarrow y = \frac{2}{3} \cdot 4\sqrt{4} = \frac{16}{3}\right)$

tan. line slope 2

passing $\left(4, \frac{16}{3}\right)$

eq. $y - \frac{16}{3} = 2(x - 4)$

8.2.




Eq. of tan. at point $(\alpha, \alpha^2)$

$$y - \alpha^2 = \underbrace{2\alpha}_{\text{slope}} (x - \alpha)$$

Since it passes through the point $(1, -3)$, we have

$$-3 - \alpha^2 = 2\alpha(1 - \alpha)$$

i.e. $\alpha^2 - 2\alpha - 3 = 0$ $\quad \therefore \alpha = -1, 3$

$(\alpha + 1)(\alpha - 3)$

Eq. of tangents.

$\alpha = -1$

$$y - (-1)^2 = 2(-1)(x - (-1))$$

i.e. $$y - 1 = -2(x+1)$$

$\alpha = 3$

$$y - 3^2 = 2.3(x-3)$$

i.e. $$y - 9 = 6(x-3)$$

9.1.

(i) $y = \sin(\sin(\sin x))$

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[rrrddd, bend right] & \sin x \arrow[d, equal] & & \\
& u \arrow[r, mapsto] & \sin u \arrow[d, equal] & \\
& & v \arrow[r, mapsto] & \sin v \arrow[d, equal] \\
& & & y
\end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \cos v \cdot \cos u \cdot \cos x$$

$$= \cos(\sin(\sin x))$$

$$\cdot \cos(\sin x)$$

$$\cdot \cos x.$$

(ii) $\quad y = \left(\dfrac{t-2}{2t+1}\right)^{9}$

$$\begin{tikzcd} t \arrow[r, mapsto] \arrow[ddrr, bend right] & \left(\dfrac{t-2}{2t+1}\right) \arrow[d, equal] & \\ & u \arrow[r, mapsto] & u^{9} \arrow[d, equal] \\ & & y \end{tikzcd}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= 9u^{8} \cdot \frac{1 \cdot (2t+1) - (t-2)\cdot 2}{(2t+1)^{2}}$$

$$= 9\left(\frac{t-2}{2t+1}\right)^{8} \cdot \frac{5}{(2t+1)^{2}}$$

(iii) $$y = \sqrt{x + \sqrt{x + \sqrt{x}}}$$

Note that

$$\frac{d}{dx}\left(\sqrt{x+u}\right) = \frac{1 + \frac{du}{dx}}{2\sqrt{x+u}}.$$

Using this formula

firstly with $u = \sqrt{x + \sqrt{x}}$

secondly with $u = \sqrt{x}$,

one obtains

$$\frac{dy}{dx} = \frac{1 + \frac{d}{dx}\left(\sqrt{x+\sqrt{x}}\right)}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

$$= \frac{1 + \dfrac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}$$

(iv) $\qquad y = e^{\sec 3\theta}$

$$
\begin{tikzcd}
\theta \arrow[r, mapsto] \arrow[rrrddd, bend right, dashed] & 3\theta \arrow[d, equal] & & \\
& u \arrow[r, mapsto] & \sec u \arrow[d, equal] & \\
& & v \arrow[r, mapsto] & e^{v} \arrow[d, equal] \\
& & & y
\end{tikzcd}
$$

$$
\frac{dy}{d\theta} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{d\theta}
$$

$$
= e^{v} \cdot \sec u \cdot \tan u \cdot 3
$$

$$
= e^{\sec 3\theta} \cdot \sec 3\theta \cdot \tan 3\theta \cdot 3
$$

(v) $$y = e^{2^{x^3}}$$

$$\begin{array}{ccccccc} x & \longmapsto & x^3 & & & & \\ & & \shortparallel & & & & \\ & & u & \longmapsto & 2^u & & \\ & & & & \shortparallel & & \\ & & & & v & \longmapsto & e^v \\ & & & & & & \shortparallel \\ & & & & & & y \end{array}$$

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= e^v \cdot \ln 2 \cdot 2^u \cdot 3x^2$$

$$= e^{2^{x^3}} \cdot \ln 2 \cdot 2^{x^3} \cdot 3x^2$$

9.2.

$$F(x) = f(x)^2 \cdot f(g(x))$$

$$F'(x) = 2\, f(x) \cdot f'(x) \cdot f(g(x)) + f(x)^2 \cdot f'(g(x)) \cdot g'(x)$$

$$F'(1) = 2\, f(1) \cdot f'(1) \cdot f(\overbrace{g(1)}^{3}) + f(1)^2 \cdot f'(\underbrace{g(1)}_{3}) \cdot g'(1)$$

$$= 2 \cdot 5 \cdot 4 \cdot (-1) \qquad -40$$

$$+ 5^2 \cdot (-2) \cdot 2 \qquad -100$$

$$= -140$$

9.3

$$x^3 = g(h(x))$$

with $h(1) = 5$ & $h'(1) = 7$.

By taking the derivative of both sides of the above equation, we conclude

$$3x^2 = g'(h(x)) \cdot h'(x)$$

$\longrightarrow$

$$3 \cdot 1^2 = g'(h(1)) \cdot h'(1)$$

$$= g'(5) \cdot 7$$

$$\therefore \quad g'(5) = \frac{3}{7}$$

10.1

(i)
$$y = x^x$$
$$\ln y = x \cdot \ln x$$
$$\frac{d}{dx}(\quad) = \frac{d}{dx}(\quad)$$
$$\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{x}$$
$$\longrightarrow \frac{dy}{dx} = y \cdot (\ln x + 1)$$
$$= x^x (\ln x + 1)$$

(ii) $$y = (\ln x)^{\tan 3x}$$

$$\ln y = \tan 3x \cdot \ln(\ln x)$$

$$\frac{d}{dx}(\quad) = \frac{d}{dx}(\quad)$$

$$\frac{1}{y}\frac{dy}{dx} = \sec^2(3x)\cdot 3 \cdot \ln(\ln x) + \tan 3x \cdot \frac{1/x}{\ln x}$$

$$\frac{dy}{dx} = y \cdot (\quad)$$

$$= (\ln x)^{\tan 3x} \cdot \left\{ 3\sec^2(3x)\cdot \ln(\ln x) + \tan 3x \cdot \frac{1}{x \ln x} \right\}.$$

(iii)

$$y = \left(\sqrt{x}\right)^{\sin x}$$

$$\ln y = \sin x \cdot \ln\left(\sqrt{x}\right)$$

$$= \frac{1}{2}\sin x \cdot \ln x$$

$$\frac{d}{dx}\Big( \qquad \Big) = \frac{d}{dx}\Big( \qquad \qquad \Big)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left\{\cos x \cdot \ln x + \sin x \cdot \frac{1}{x}\right\}$$

$$\frac{dy}{dx} = \frac{1}{2}y\left\{\cos x \cdot \ln x + \frac{\sin x}{x}\right\}$$

$$= \frac{1}{2}\left(\sqrt{x}\right)^{\sin x}\left\{\cos x \cdot \ln x + \frac{\sin x}{x}\right\}$$

(iv)

$$y = x^{\ln x}$$

$$\ln y = \ln x \cdot \ln x$$

$$= (\ln x)^2$$

$$\frac{d}{dx}(\qquad) = \frac{d}{dx}(\qquad)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \ln x \cdot \frac{1}{x} = \frac{2 \ln x}{x}$$

$$\frac{dy}{dx} = y \cdot \frac{2 \ln x}{x}$$

$$= x^{\ln x} \cdot \frac{2 \ln x}{x}$$

$$= 2 x^{\ln x - 1} \cdot \ln x$$

(v) $$y = (\cot x)^{\sin x}$$

$$\ln y = \sin x \cdot \ln(\cot x)$$

$$= \sin x \,\{\ln(\cos x) - \ln(\sin x)\}$$

$$\frac{d}{dx}(\quad) = \frac{d}{dx}(\quad)$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \cdot \{\ln(\cos x) - \ln(\sin x)\}$$

$$+ \sin x \cdot \left\{ \frac{-\sin x}{\cos x} - \frac{\cos x}{\sin x} \right\}$$

$$= \cos x \cdot \ln(\cot x)$$

$$- \sin x \cdot \tan x - \sin x \cdot \cot x$$

$$\frac{dy}{dx} = y \cdot \{\qquad\qquad\}$$

$$= (\cot x)^{\sin x} \cdot$$

$$\{\cos x \cdot \ln(\cot x) - \sin x \cdot \tan x$$

$$- \sin x \cdot \cot x\}$$

11.1

$$f(x) + x^2 \cdot \{f(x)\}^3 = 10.$$

$$\frac{d}{dx}\left( \qquad \right) = \frac{d}{dx}\left( \qquad \right)$$

$$f'(x) + 2x \cdot \{f(x)\}^3$$
$$+ x^2 \cdot 3\{f(x)\}^2 \cdot f'(x) = 0.$$

$$f'(1) + 2 \cdot 1 \cdot \{f(1)\}^3$$
$$+ 1^2 \cdot 3\{f(1)\}^2 \cdot f'(1) = 0.$$

i.e.

$$f'(1) + 2 \cdot 1 \cdot 2^3$$
$$+ 1^2 \cdot 3 \cdot 2^2 \cdot f'(1) = 0$$

$$13 \cdot f'(1) + 16 = 0.$$

$$f'(1) = -\frac{16}{13}.$$

11.2.

$$x^2 + 2xy - y^2 + x = 2.$$

$$\frac{d}{dx}\left( \qquad \right) = \frac{d}{dx}\left( \quad \right)$$

$$2x + 2 \cdot y + 2x \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} + 1 = 0.$$

at $(x, y) = (1, 2)$, we have

$$2 \cdot 1 + 2 \cdot 2 + 2 \cdot 1 \cdot \frac{dy}{dx} - 2 \cdot 2 \cdot \frac{dy}{dx} + 1 = 0$$

i.e.

$$-2 \cdot \frac{dy}{dx} = -7$$

$$\longrightarrow \quad \frac{dy}{dx} = \frac{7}{2}.$$

11.3

$$e^{\frac{x}{y}} = 7x - y$$

$$\frac{d}{dx}\left(\quad\right) = \frac{d}{dx}\left(\quad\quad\right)$$

$$e^{\frac{x}{y}}\left(\frac{1\cdot y - x\cdot\frac{dy}{dx}}{y^2}\right) = 7 - \frac{dy}{dx}$$

$$\left(1 - \frac{x}{y^2}\cdot e^{\frac{x}{y}}\right)\frac{dy}{dx} = 7 - \frac{1}{y}e^{\frac{x}{y}}$$

$$\frac{dy}{dx} = \frac{7 - \frac{e^{\frac{x}{y}}}{y}}{1 - \frac{x\cdot e^{\frac{x}{y}}}{y^2}}$$

$$= \frac{7y^2 - y\cdot e^{\frac{x}{y}}}{y^2 - x\cdot e^{\frac{x}{y}}}$$

11.4.

$$\sqrt{y} + \sqrt{x} = 3$$

$$\frac{d}{dx}\left( \qquad \right) = \frac{d}{dx}\left( \qquad \right)$$

$$\frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} + \frac{1}{2\sqrt{x}} = 0$$

At $(1, 4)$, we have

$$\frac{1}{2\sqrt{4}} \cdot \frac{dy}{dx} + \frac{1}{2\sqrt{1}} = 0.$$

$$\longrightarrow \quad \frac{dy}{dx} = -2$$

Eq. of tan.

$$y - 4 = (-2)(x - 1)$$

i.e.

$$y = -2x + 6$$

12.1

$$f(x) = e^x \quad \text{and} \quad a = 0$$

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= e^0 + e^0(x-0) \\ &= 1 + 1 \cdot (x-0) \\ &= 1 + x \end{aligned}$$

$$\begin{aligned} e^{0.01} &= f(0.01) \\ &\approx L(0.01) \\ &= 1 + 0.01 = 1.01 \end{aligned}$$

12.2.

Set $f(x) = \sqrt[3]{x}$.

Consider the linear approximation $L(x)$ of $f(x) = \sqrt[3]{x}$ at $a = 27$.

$$L(x) = f(a) + f'(a)(x-a)$$

$$= \sqrt[3]{27} + \frac{1}{3}\frac{1}{(\sqrt[3]{27})^2}(x-27)$$

$$= 3 + \frac{1}{3}\cdot\frac{1}{3^2}(x-27)$$

$$= 3 + \frac{1}{27}(x-27)$$

$$\sqrt[3]{26.8} = f(26.8)$$

$$\approx L(26.8)$$

$$= 3 + \frac{1}{27}(26.8-27)$$

$$= 3 - \frac{1}{27}\times 0.2$$

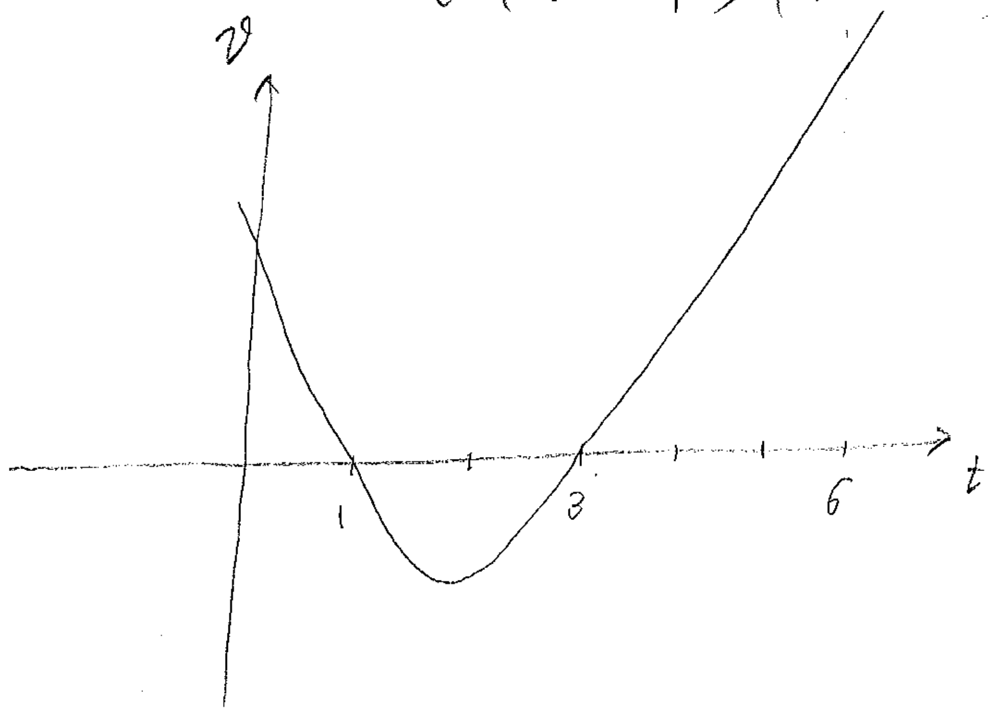
13.2

$$s = f(t) = t^3 - 6t^2 + 9t$$

$$v = f'(t) = 3t^2 - 12t + 9$$

$$= 3(t^2 - 4t + 3)$$

$$= 3(t-1)(t-3)$$




| $t$ | 0 | | 1 | | 3 | | 6 |
|---|---|---|---|---|---|---|---|
| $v$ | | + | 0 | − | 0 | + | |

$$\underset{f(0)=0}{\overset{t=0}{\bullet}} \xrightarrow{\quad 4 \quad} \underset{f(1)=4}{\overset{t=1.}{\bullet}}$$

$$\underset{f(3)=0}{\overset{t=3}{\bullet}} \xleftarrow{\quad 4. \quad} \underset{f(1)=4}{\overset{t=1.}{\bullet}}$$

$$\underset{f(3)=0}{\overset{t=3}{\bullet}} \xrightarrow{\quad 54 \quad} \underset{f(6)=54}{\overset{t=6}{\bullet}}$$

total distance traveled

$$= 4 + 4 + 54 = 62.$$

13.3.

$$h(t) = 48t - 16t^2$$

$$h'(t) = 48 - 16 \cdot 2t$$

$$= 48 - 32t$$

$$h(t) = 48t - 16t^2 = 32$$

$$-16t^2 + 48t - 32 = 0$$

$$\shortparallel$$

$$-16(t^2 - 3t + 2)$$

$$\shortparallel$$

$$-16(t-1)(t-2)$$

$$t = 1, 2$$

On its way up $\longrightarrow$ $t = 1$.

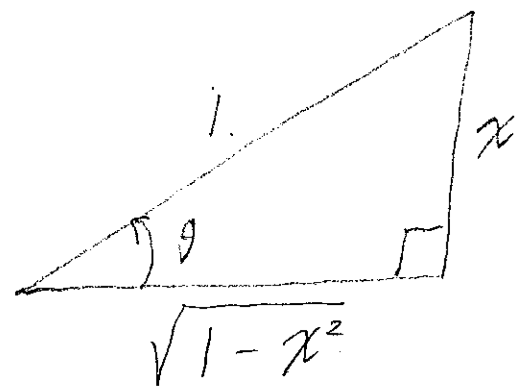
$$h'(1) = 48 - 32 \cdot 1 = 16$$

14. 1.

(i) $\tan(2\sin^{-1}x)$

Set $\theta = \sin^{-1}x$

$\longrightarrow \sin\theta = x$




$\tan(2\sin^{-1}x)$

$= \tan(2\theta)$

$$= \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

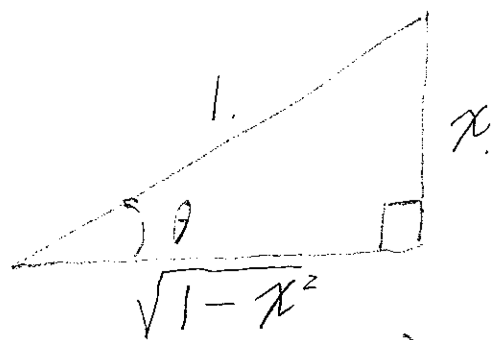
$$= \frac{2 \cdot \frac{x}{1} \cdot \frac{\sqrt{1-x^2}}{1}}{\left(\frac{\sqrt{1-x^2}}{1}\right)^2 - \left(\frac{x}{1}\right)^2}$$

$$= \frac{2x\sqrt{1-x^2}}{1-2x^2}$$

(ii) $\cos(2\sin^{-1}x)$

Set $\theta = \sin^{-1}x$

$\rightarrow \sin\theta = x$




$\cos(2\sin^{-1}x)$

$= \cos(2\theta)$

$= \cos^2\theta - \sin^2\theta$

$= \left(\frac{\sqrt{1-x^2}}{1}\right)^2 - \left(\frac{x}{1}\right)^2$

$= 1 - 2x^2$

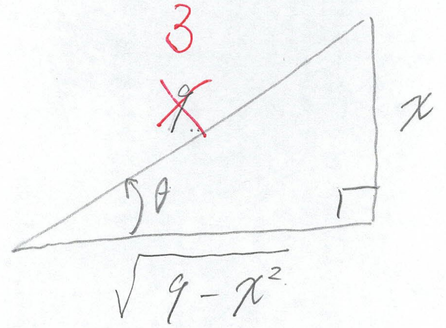
14.1 (iii)

48

$$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right)$$

Let $\theta = \tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)$

$$\longrightarrow \tan\theta = \frac{x}{\sqrt{9-x^2}}$$




$$\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right)$$

$$= \cos\theta$$

$$= \frac{\sqrt{9-x^2}}{\cancel{9}\ 3}$$

15.1.

(i) $\sinh(0)$

$$= \frac{e^0 - e^{-0}}{2} = 0$$

(ii) $\sinh(\ln 5)$

$$= \frac{e^{\ln 5} - e^{-\ln 5}}{2}$$

$$= \frac{5 - \frac{1}{5}}{2} = \frac{12}{5}$$

(iii) $\dfrac{1 + \tanh(1/2)}{1 - \tanh(1/2)}$

$$= \frac{1 + \dfrac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}}{1 - \dfrac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}}$$

$$= \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}} + e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}} - (e^{\frac{1}{2}} - e^{-\frac{1}{2}})} = \frac{2e^{\frac{1}{2}}}{2e^{-\frac{1}{2}}}$$

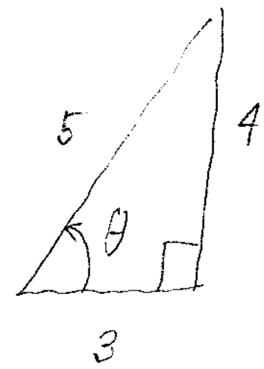
$$= e.$$

17.1.

(i) $\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(\frac{4}{5}\right)$

$\rightarrow \sin\theta = \frac{4}{5}$.

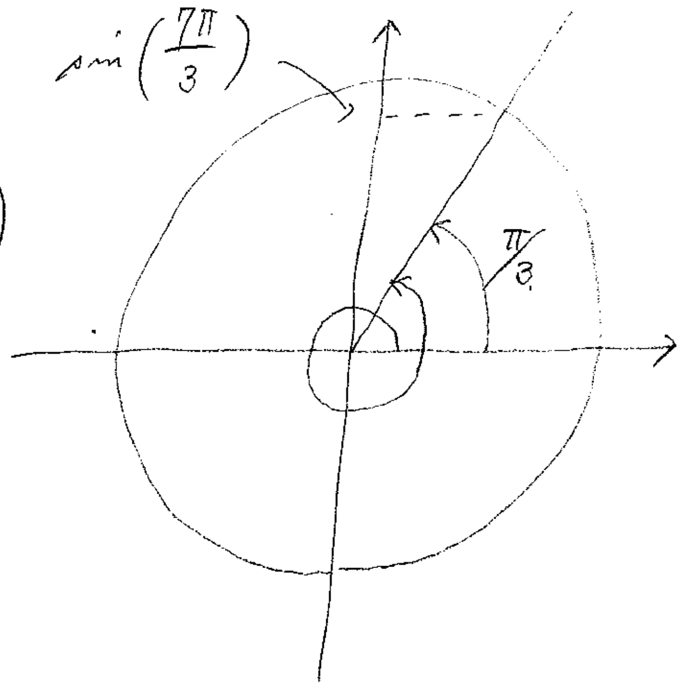



$\tan\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

$= \tan\theta$

$= \frac{4}{3}$

(ii)



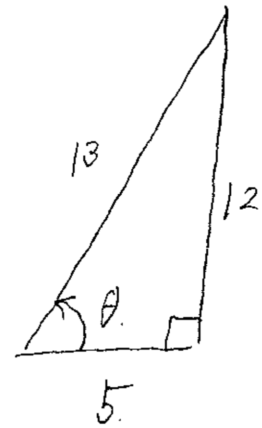

$\sin^{-1}\left(\sin\left(\frac{7\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

(iii) $$\sin\left(2\sin^{-1}\left(\frac{12}{13}\right)\right)$$

Let $$\theta = \sin^{-1}\left(\frac{12}{13}\right)$$

$$\rightarrow \sin\theta = \frac{12}{13}$$




$$\sin\left(2\sin^{-1}\left(\frac{12}{13}\right)\right)$$

$$= \sin(2\theta)$$

$$= 2\sin\theta\cos\theta$$

$$= 2\cdot\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) = \frac{120}{169}$$

18.1.

$768 - 1024 + 288$

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

over $[-1, 4]$

$$f'(x) = 12x^3 - 48x^2 + 36x$$

$$= 12x(x^2 - 4x + 3)$$

$$= 12x(x-1)(x-3)$$

| $x$ | $-1$ | | $0$ | | $1$ | | $3$ | | $4$ |
|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $+$ | |
| $f(x)$ | $37$ | $\searrow$ | $0$ | $\nearrow$ | $5$ | $\searrow$ | $-27$ | $\nearrow$ | $32$ |
| | abs. max | | local min | | local max | | local min, abs. min | | |

abs. max $37$ at $x = -1$.

abs. min $-27$ at $x = 3$

local max $5$ at $x = 1$.

local min $0$ at $x = 0$.

$-27$ at $x = 3$

$16 \times 16 = 256$, $256 \times 3 = 768$, $64 \times 16 = 1024$, $18 \times 16 = 288$, $768 + 288 = 1056$

18.2

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

Step 1 (i) $a = -2$, $b = 3$

(ii)

$$f'(x) = 6x^2 - 6x - 12$$
$$= 6(x^2 - x - 2)$$
$$= 6(x+1)(x-2)$$

$f'(c) = 0$ $c = -1, 2$

Step 2.

$f(-2) = -3$

$f(3) = -8$

$f(-1) = 8$ $\leftarrow$ abs. max.

$f(2) = -19$ $\leftarrow$ abs. min.

(b) $f(x) = x e^{\frac{x}{2}}$ on $[-3, 1]$

Step 1

(i) $a = -3$ , $b = 1$.

(ii) $$f'(x) = 1 \cdot e^{\frac{x}{2}} + x \cdot \frac{1}{2} e^{\frac{x}{2}}$$
$$= e^{\frac{x}{2}} \left( 1 + \frac{x}{2} \right)$$

$f'(c) = 0$ $\qquad$ $c = -2$.

Step 2.

$f(-3) = -3 e^{-\frac{3}{2}}$

$f(1) = e^{\frac{1}{2}}$ $\leftarrow$ abs. max

$f(-2) = -2 e^{-1}$ $\leftarrow$ abs. mini

| $x$ | $-3$ | | $-2$ | | $1$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | | $+$ | |
| $f(x)$ | | $\searrow$ | | $\nearrow$ | |

(c) $f(x) = x\sqrt{32 - x^2}$ on $[0, 5]$

Step 1.

(i) $a = 0$, $b = 5$

(ii)

$$f'(x) = 1 \cdot \sqrt{32 - x^2} + x \cdot \frac{-2x}{2\sqrt{32 - x^2}}$$

$$= \frac{32 - 2x^2}{\sqrt{32 - x^2}}$$

$$= \frac{-2(x^2 - 16)}{\sqrt{32 - x^2}}$$

$f'(c) = 0$ $\qquad c = \cancel{\pm} 4$

$c = 4$

Step 2.

$f(0) = 0 \quad \leftarrow$ abs. min

$f(5) = 5\sqrt{7}$

$f(4) = 16 \quad \leftarrow$ abs. max

(d) $f(t) = 2\cos t + \sin 2t$ on $[0, \frac{\pi}{2}]$

Step 1.

(i) $a = 0$, $b = \frac{\pi}{2}$

(ii) $f'(t) = -2\sin t + 2\cos 2t$

$= -2\sin t + 2(1 - 2\sin^2 t)$

$= -2(2\sin^2 t + \sin t - 1)$

$= -2(2\sin t - 1)(\sin t + 1)$

$f'(c) = 0 \iff 2\sin c - 1 = 0$

$c = \frac{\pi}{6}$

Step 2.

$f(0) = 2.$

$f(\frac{\pi}{2}) = 0 \quad \leftarrow$ abs. min

$f(\frac{\pi}{6}) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$\leftarrow$ abs. max

(e) $f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$

Step 1. (i) $a = -1$, $b = 1$.

(ii) $$f'(x) = \frac{2x+1}{x^2+x+1}$$

$f'(c) = 0$ $\quad c = -\frac{1}{2}$.

Step 2.

$f(-1) = \ln 1 = 0$

$f(1) = \ln 3$ ← abs. max

$f(-\frac{1}{2}) = \ln\left(\frac{3}{4}\right)$ ← abs. min

19.1.

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5)$$

| $x$ | | $-2$ | | $-1$ | | $1$ | | $3$ | | $5$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $0$ | $-$ | $0$ | $+$ | $0$ | $-$ | $0$ | $-$ | $0$ | $+$ |
| $f(x)$ | $\searrow$ | | $\searrow$ | | $\nearrow$ | | $\searrow$ | | $\searrow$ | | $\nearrow$ |
| | | | | local min | | local max | | | | local min | |

(a) local max $\quad x = 1$.

(b) local min $\quad x = -1, 5$.

19.2.

$$f(x) = x^8 (x-4)^7$$

$$f'(x) = 8x^7 (x-4)^7 + x^8 \cdot 7(x-4)^6$$

$$= x^7 (x-4)^6 \{ 8(x-4) + x \cdot 7 \}$$

$$= x^7 (x-4)^6 (15x - 32)$$

$$f''(x) = 7x^6 (x-4)^6 (15x-32) + x^7 \cdot 6(x-4)^5 (15x-32) + x^7 \cdot (x-4)^6 \cdot 15$$

$$= x^6 (x-4)^5 \{ 7 \cdot (x-4)(15x-32) + x \cdot 6 \cdot (15x-32) + x \cdot (x-4) \cdot 15 \}$$

| $x$ | | $0$ | | $\frac{32}{15}$ | | $4$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $-$ | $0$ | $+$ | $0$ | $+$ |
| $f''(x)$ | | $0$ | | $+$ | | $0$ | |
| $f(x)$ | | | | | | | |

1st der. test: local max, local min, neither

2nd der. test: inconclusive, local min, inconclusive

20.1

(a) $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$ $\left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{1/x}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \dfrac{2\sqrt{x}}{x}$

$= \lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

(b) $\lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$ $\left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\sin x}{6x}$ $\left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$

(c) $\lim_{x\to 0} \dfrac{\sin x}{1-x^2} = 0$

$\sin x \to 0$

$1-x^2 \to 1$

(d) $\displaystyle\lim_{x \to 0} \frac{7^x - 6^x}{3^x - 2^x} \qquad \left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\ln 7 \cdot 7^x - \ln 6 \cdot 6^x}{\ln 3 \cdot 3^x - \ln 2 \cdot 2^x}$$

$$= \frac{\ln 7 - \ln 6}{\ln 3 - \ln 2}.$$

21.1.

(a) $$\lim_{x\to 0^+} \underbrace{\sin x}_{\downarrow \atop 0} \cdot \underbrace{\ln(\underbrace{2x}_{\downarrow \atop 0^+})}_{\downarrow \atop (-\infty)}$$

$$0 \times (-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln(2x)}{\frac{1}{\sin x}} \qquad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{2x}\cdot 2}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} - \underbrace{\frac{1}{\cos x}}_{\swarrow \atop 1} \cdot \frac{\sin x}{\underbrace{x}_{\to 0}} \cdot \overbrace{\sin x}^{\to 0.}$$

$$= 0.$$

(b) $\lim_{x\to\infty} \underbrace{\underbrace{2x}_{\downarrow\ \infty} \cdot \tan\left(\underbrace{\frac{1}{3x}}_{\downarrow\ 0}\right)}_{\downarrow}$

$\times \quad 0$

$$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{3x}\right)}{\frac{1}{2x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{3x}\right)\cdot\frac{1}{3}\left(-\frac{1}{x^2}\right)}{\frac{1}{2}\left(-\frac{1}{x^2}\right)}$$

$$= \frac{2}{3}.$$

(c) $\lim_{x\to 0^+} \underbrace{\cos x}_{\downarrow\ 1} \cdot \underbrace{\arctan\left(\underbrace{\frac{3}{x}}_{\downarrow\ +\infty}\right)}_{\downarrow\ \frac{\pi}{2}} = \frac{\pi}{2}.$

(d) $\lim_{x \to \infty} \left( \sqrt{x^2+2x+3} - x \right) \quad (\infty - \infty)$

$$= \lim_{x \to \infty} \frac{\left(\sqrt{x^2+2x+3} - x\right)\left(\sqrt{x^2+2x+3} + x\right)}{\sqrt{x^2+2x+3} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2+2x+3) - x^2}{\sqrt{x^2+2x+3} + x}$$

$$= \lim_{x \to \infty} \frac{2x+3}{\sqrt{x^2+2x+3} + x}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{3}{x}}{\sqrt{1 + \frac{2}{x} + \frac{3}{x^2}} + 1}$$

$$= \frac{2}{1+1} = 1$$

(e) $\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) \quad (\infty - \infty)$

$= \lim_{x \to 0^+} \frac{\sin x - x}{x \sin x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0^+} \frac{\cos x - 1}{1 \cdot \sin x + x \cdot \cos x} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0^+} \frac{-\sin x}{\cos x + 1 \cdot \cos x + x(-\sin x)}$

$= \lim_{x \to 0^+} - \frac{\sin x}{2 \cos x} = 0$

22.1.

(a) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x}$ $\qquad (1^{\infty})$

Set $y = \left(1+\frac{3}{x}\right)^{7x}$

$$\ln y = 7x\cdot\ln\left(1+\frac{3}{x}\right)$$

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\underbrace{7x}_{\downarrow\,\infty}\cdot\underbrace{\ln\left(\underbrace{1+\frac{3}{x}}_{\downarrow\,1}\right)}_{\downarrow\,0}$$

$\times$

$$= \lim_{x\to\infty}\frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{7x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty}\frac{\frac{1}{1+\frac{3}{x}}\cdot 3\cdot\left(-\cancel{\frac{1}{x^2}}\right)}{\frac{1}{7}\left(-\frac{1}{x^2}\right)}$$

$$= 21$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y \to 21} = e^{21}.$$

(b) $\lim_{x\to\infty} \left(\frac{2x+1}{2x-1}\right)^{4x+1}$ $\quad (1^{\infty})$

Set $y = \left(\frac{2x+1}{2x-1}\right)^{4x+1}$.

$$\ln y = (4x+1)\ln\left(\frac{2x+1}{2x-1}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \underbrace{(4x+1)}_{\infty} \times \underbrace{\ln\left(\underbrace{\frac{2x+1}{2x-1}}_{1}\right)}_{0}$$

$$= \lim_{x\to\infty} \frac{\overbrace{\ln\left(\frac{2x+1}{2x-1}\right)}^{\ln(2x+1)-\ln(2x-1)}}{\frac{1}{4x+1}} \quad \left(\frac{0}{0}\right).$$

$$= \lim_{x\to\infty} \frac{\frac{2}{2x+1} - \frac{2}{2x-1}}{-\frac{4}{(4x+1)^2}}$$

$$= \lim_{x\to\infty} \frac{2 \cdot \dfrac{(2x-1)-(2x+1)}{(2x+1)(2x-1)}}{-\dfrac{4}{(4x+1)^2}}$$

$$= \lim_{x\to\infty} \frac{\overbrace{(4x+1)^2}^{16x^2 + \text{lower}}}{\underbrace{(2x+1)(2x-1)}_{4x^2 + \text{lower}}} = 4$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y \to 4} = e^4$$

(c) $\lim_{x\to\infty} (2x + e^{5x})^{\frac{1}{x}}$ $(\infty^0)$

Sea $y = (2x + e^{5x})^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln (2x + e^{5x})$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x + e^{5x})}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2x + e^{5x}} \cdot (2 + 5 \cdot e^{5x})}{1}$$

$$= \lim_{x\to\infty} \frac{2 + 5 \cdot e^{5x}}{2x + e^{5x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{5 \cdot 5 e^{5x}}{2 + 5 \cdot e^{5x}}$$

$$= \lim_{x\to\infty} \frac{25}{\frac{2}{e^{5x}} + 5} = 5$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y \to 5} = e^5$$

(d) $\lim_{x\to 0^+} \tan(5x)^{\sin x}$ $\quad (0^0)$

Set $y = \tan(5x)^{\sin x}$

$\ln y = \sin x \cdot \ln\{\tan(5x)\}$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \underbrace{\sin x}_{\downarrow \atop 0} \cdot \underbrace{\ln\{\underbrace{\tan(5x)}_{\downarrow \atop 0^+}\}}_{\downarrow \atop \times\ (-\infty)}$$

$$= \lim_{x\to 0^+} \frac{\ln\{\tan(5x)\}}{\frac{1}{\sin x}} \quad \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\tan(5x)} \sec^2(5x) \cdot 5}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} - \frac{\cos(5x)}{\cos x} \cdot \sec^2(5x) \cdot 5 \cdot \frac{\sin^2 x}{\sin(5x)}$$

$\nearrow 5 \qquad \swarrow 0$

$= 0$

see the next page

$$\lim_{x\to 0^+} \frac{\sin^2 x}{\sin(5x)} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0^+} \frac{2\sin x \cdot \cos x}{\cos(5x)\cdot 5} = 0.$$

$$\lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\overset{\to 0}{\ln y}} = e^0 = 1.$$

23.1

(a) $$y = f(x) = \frac{x}{x^2-16}$$

Domain $x \neq \pm 4$

$$f'(x) = \frac{1\cdot(x^2-16) - x\cdot 2x}{(x^2-16)^2}$$

$$= \frac{-(x^2+16)}{(x^2-16)^2}$$

$$f''(x) = \frac{(-2x)(x^2-16)^2 + (x^2+16)2(x^2-16)\cdot 2x}{(x^2-16)^4}$$

$$= \frac{(-2x)(x^2-16) + (x^2+16)4x}{(x^2-16)^3}$$

$$= \frac{2x(x^2+48)}{(x^2-16)^3}$$

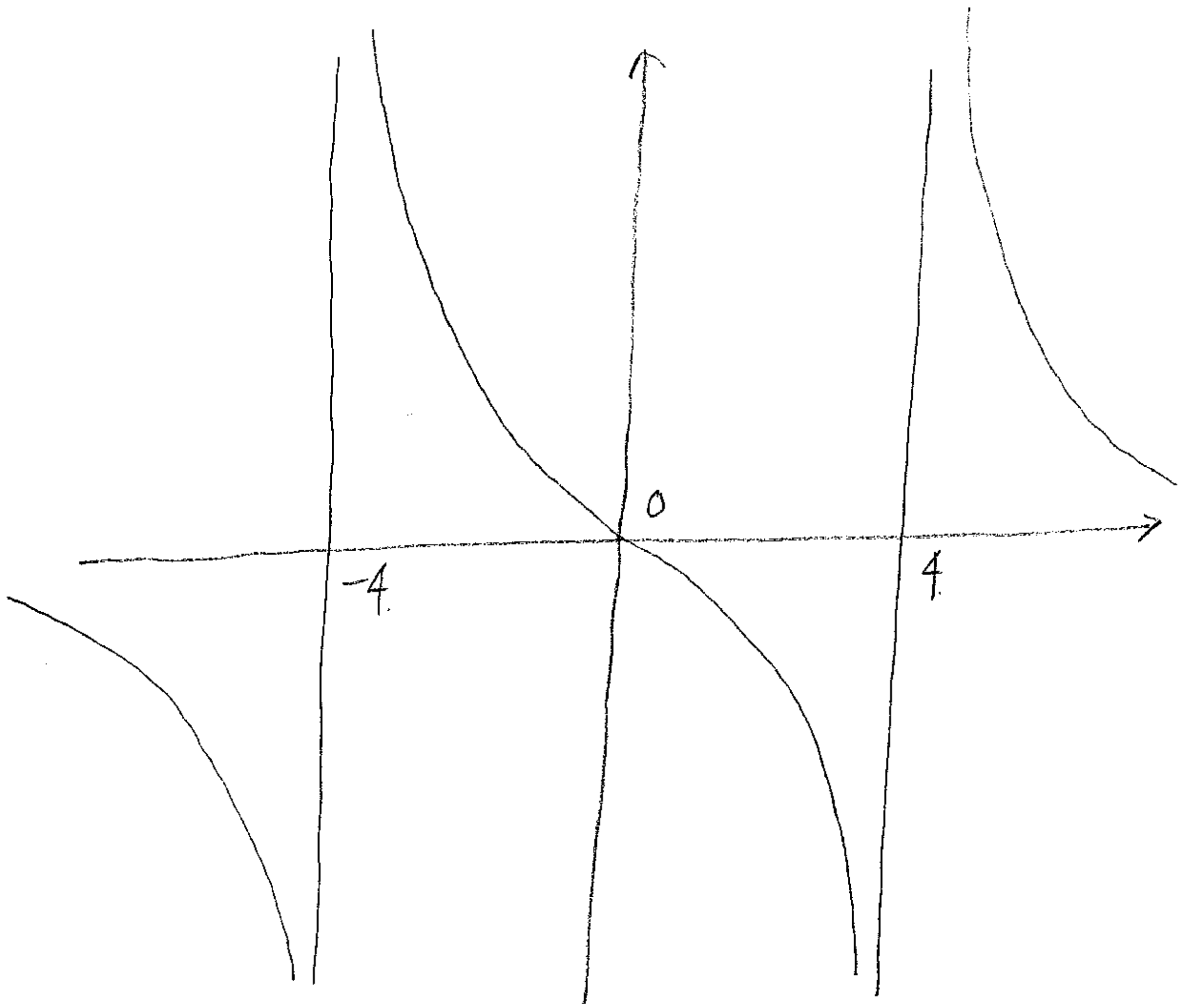
| $x$ | | $-4$ | | $0$ | | $4$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | × | $-$ | $-$ | $-$ | × | $-$ |
| $f''(x)$ | $-$ | × | $+$ | $0$ | $-$ | × | $+$ |
| $f(x)$ | | × | | | | × | |

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{x}{x^2-16} = -\infty$$

$$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \frac{x}{x^2-16} = +\infty$$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{x}{x^2-16} = -\infty$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{x}{x^2-16} = +\infty$$



0
-4
4

(b) $\quad y = f(x) = \dfrac{x}{x^2+16}$

Domain $(-\infty, \infty)$

$$f'(x) = \frac{1\cdot(x^2+16) - x\cdot 2x}{(x^2+16)^2}$$

$$= \frac{-(x^2-16)}{(x^2+16)^2}$$

$$= \frac{-(x+4)(x-4)}{(x^2+16)^2}$$

$$f''(x) = \frac{(-2x)(x^2+16)^2 + (x^2-16)2(x^2+16)\cdot 2x}{(x^2+16)^4}$$

$$= \frac{(-2x)(x^2+16) + (x^2-16)4x}{(x^2+16)^3}$$

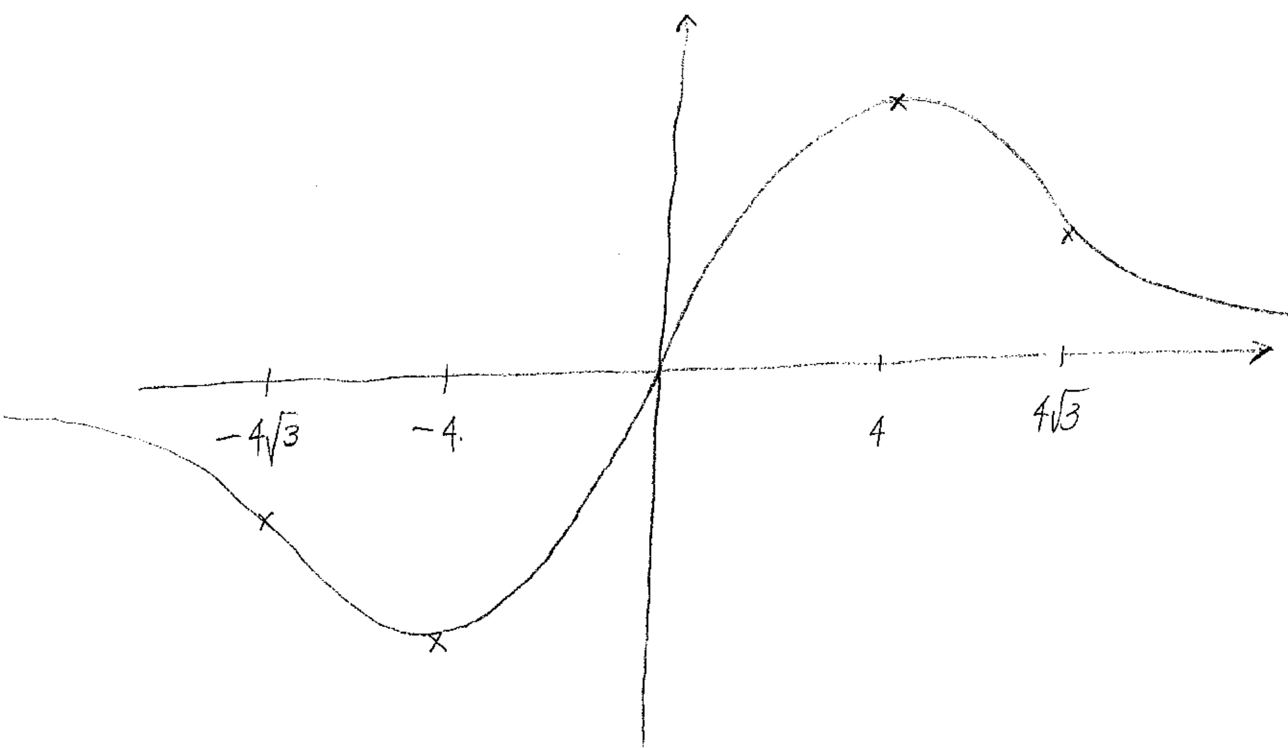
$$= \frac{2x^3 - 96x}{(x^2+16)^3}$$

$$= \frac{2x(x+4\sqrt{3})(x-4\sqrt{3})}{(x^2+16)^3}$$

| $x$ | | $-4\sqrt{3}$ | | $-4$ | | $0$ | | $4$ | | $4\sqrt{3}$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $-$ | $-$ | $0$ | $+$ | $+$ | $+$ | $0$ | $-$ | $-$ | $-$ |
| $f''(x)$ | $-$ | $0$ | $+$ | $+$ | $+$ | $0$ | $-$ | $-$ | $-$ | $0$ | $+$ |
| $f(x)$ | | | | | | | | | | | |

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+16} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x^2+16} = 0$$



(c) $y = f(x) = \dfrac{1}{x^2 - 16}$

Domain $x \neq \pm 4$

$$f'(x) = \frac{-2x}{(x^2-16)^2}$$

$$f''(x) = \frac{-2(x^2-16)^2 + 2x \cdot 2(x^2-16) \cdot 2x}{(x^2-16)^4}$$

$$= \frac{-2(x^2-16) + 8x^2}{(x^2-16)^3}$$

$$= \frac{6x^2 + 32}{(x^2-16)^3}$$

| $x$ | | $-4$ | | $0$ | | $4$ | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $+$ | X | $+$ | $0$ | $-$ | X | $-$ |
| $f''(x)$ | $+$ | X | $-$ | $-$ | $-$ | X | $+$ |
| $f(x)$ | | X | | | | X | |

$$\lim_{x\to\infty} \frac{1}{x^2-16} = 0$$

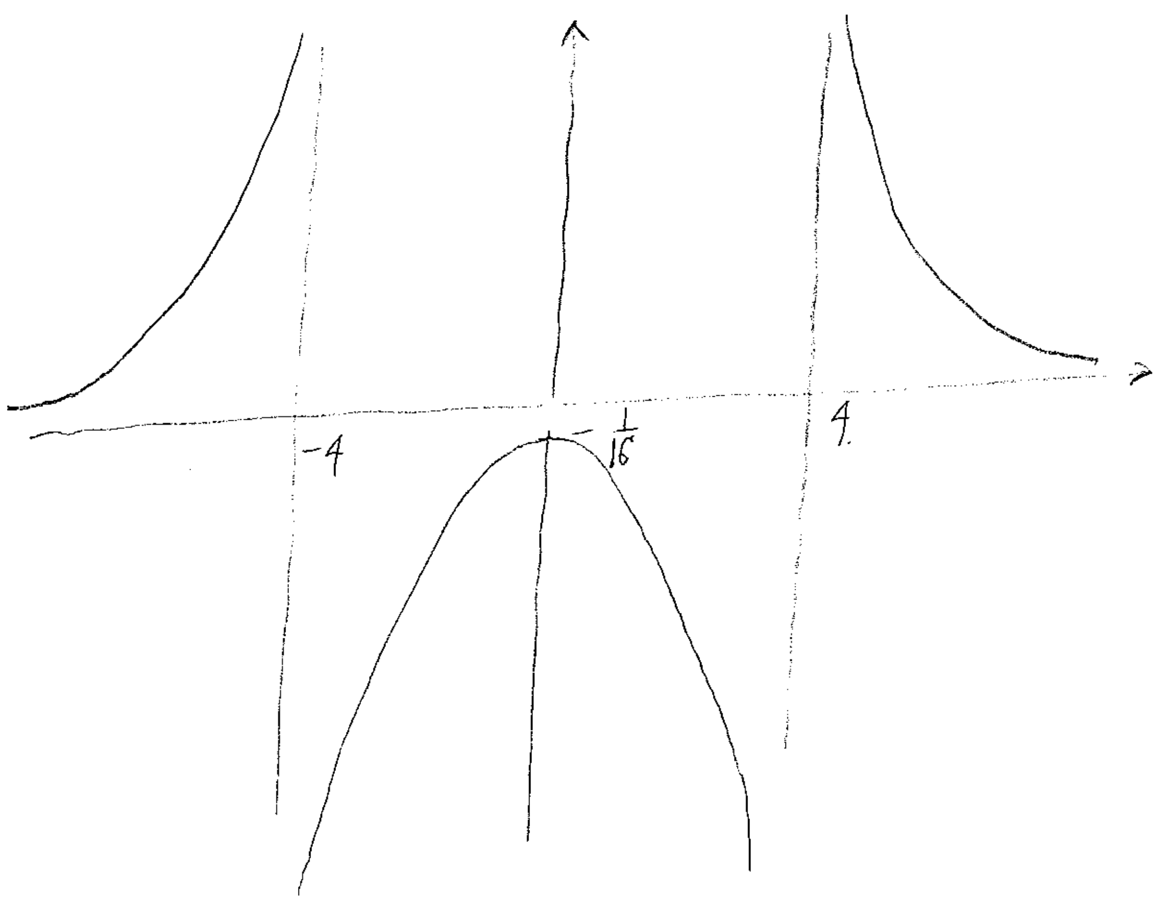
$$\lim_{x\to-\infty} \frac{1}{x^2-16} = 0$$

$$\lim_{x\to-4^-} \frac{1}{x^2-16} = +\infty$$

$$\lim_{x\to-4^+} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^-} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^+} \frac{1}{x^2-16} = +\infty$$



(d) $y = f(x) = \dfrac{x^2}{x^2 - 16}$

Domain $x \neq \pm 4$

$$f'(x) = \frac{2x(x^2-16) - x^2 \cdot 2x}{(x^2-16)^2}$$

$$= \frac{-32x}{(x^2-16)^2}$$

$$f''(x) = \frac{-32(x^2-16)^2 + 32x \cdot 2(x^2-16) \cdot 2x}{(x^2-16)^4}$$

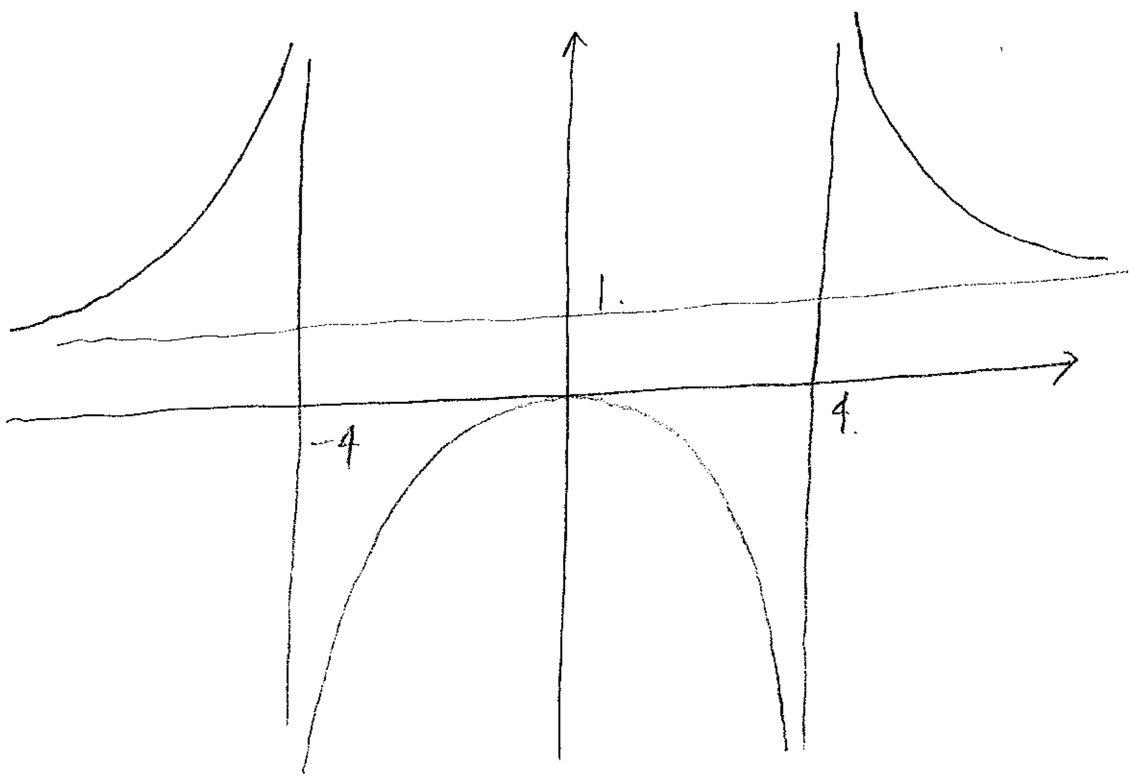
$$= \frac{-32\{(x^2-16) - 4x^2\}}{(x^2-16)^3}$$

$$= \frac{32(3x^2+16)}{(x^2-16)^3}$$

| $x$ | | $-4$ | | $0$ | | $4$ | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| $f'(x)$ | $+$ | X | $+$ | $0$ | $-$ | X | $-$ |
| $f''(x)$ | $+$ | X | $-$ | $-$ | $-$ | X | $+$ |
| $f(x)$ | | X | | $0$ | | X | |

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-16} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2-16} = 1$$



(e) $\quad y = f(x) = e^{-x}\sin x$

on $[0, 2\pi]$

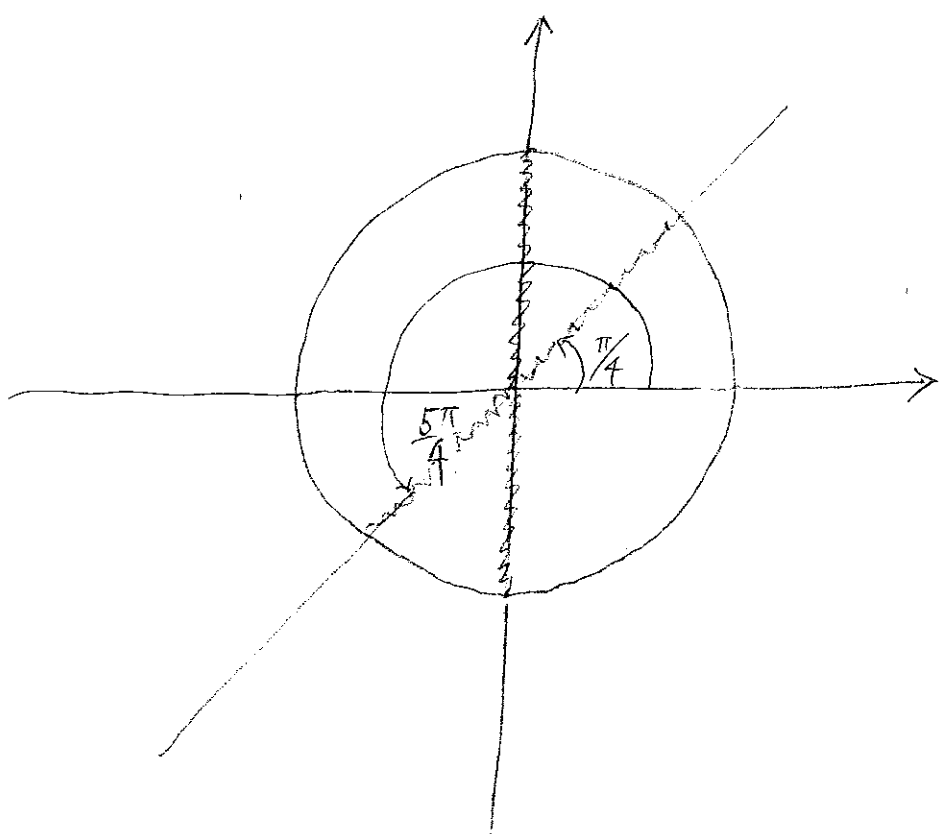
$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$$

$$= e^{-x}(\cos x - \sin x)$$

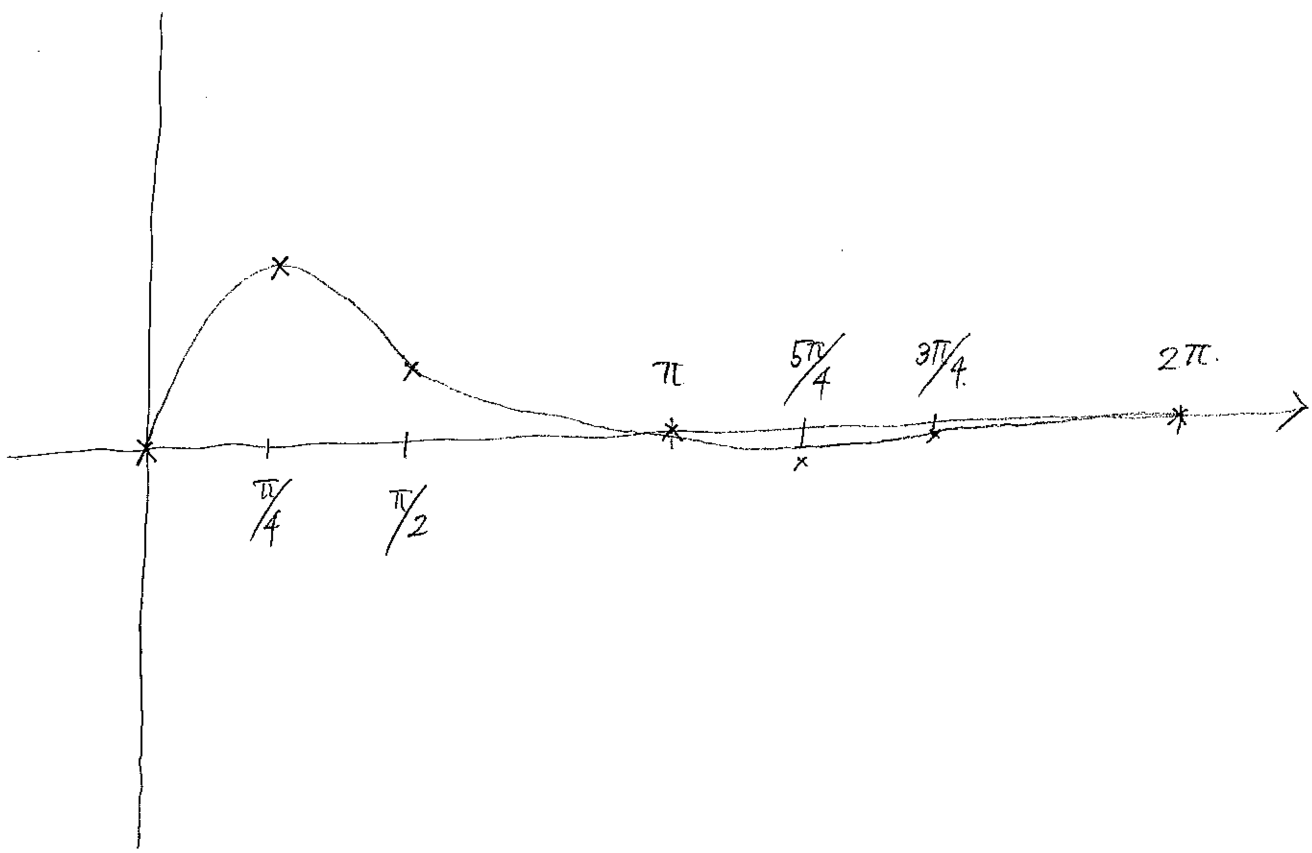
$$f''(x) = \cancel{e^{-x}\sin x} - e^{-x}\cos x$$

$$- e^{-x}\cos x \cancel{- e^{-x}\sin x}$$

$$= -2e^{-x}\cos x$$



| $x$ | $0$ | | $\pi/4$ | | $\pi/2$ | | $5\pi/4$ | | $3\pi/2$ | | $2\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $+$ | $0$ | $-$ | $-$ | $-$ | $0$ | $+$ | $+$ | $+$ | |
| $f''(x)$ | | $-$ | $-$ | $-$ | $0$ | $+$ | $+$ | $+$ | $0$ | $-$ | $-$ |
| $f(x)$ | $0$ | $\nearrow$ | $e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}$ | $\searrow$ | $e^{-\frac{\pi}{2}}$ | $\searrow$ | $-e^{-\frac{5\pi}{4}}\cdot\frac{\sqrt{2}}{2}$ | $\nearrow$ | $-e^{-\frac{3\pi}{2}}$ | $\nearrow$ | $0$ |



23. 1. (f)

$$y = f(x) = \ln|x^2 - 10x + 24|$$

Condition

$$|x^2 - 10x + 24| > 0$$

i.e. $x^2 - 10x + 24 \neq 0$.

$$\underset{(x-4)(x-6)}{\underbrace{x^2 - 10x + 24}}$$

i.e. $x \neq 4, 6$.

Domain

$$(-\infty, 4) \cup (4, 6) \cup (6, \infty)$$

$$f'(x) = \frac{2x - 10}{x^2 - 10x + 24}$$

$$= \frac{2(x-5)}{(x-4)(x-6)}$$

$$f''(x) = \frac{2\cdot(x^2-10x+24)-(2x-10)\cdot(2x-10)}{(x^2-10x+24)^2}$$

$$= \frac{-2x^2+20x-52}{(x^2-10x+24)^2}$$

$$= \frac{-2(x^2-10x+26)}{(x^2-10x+24)^2}$$

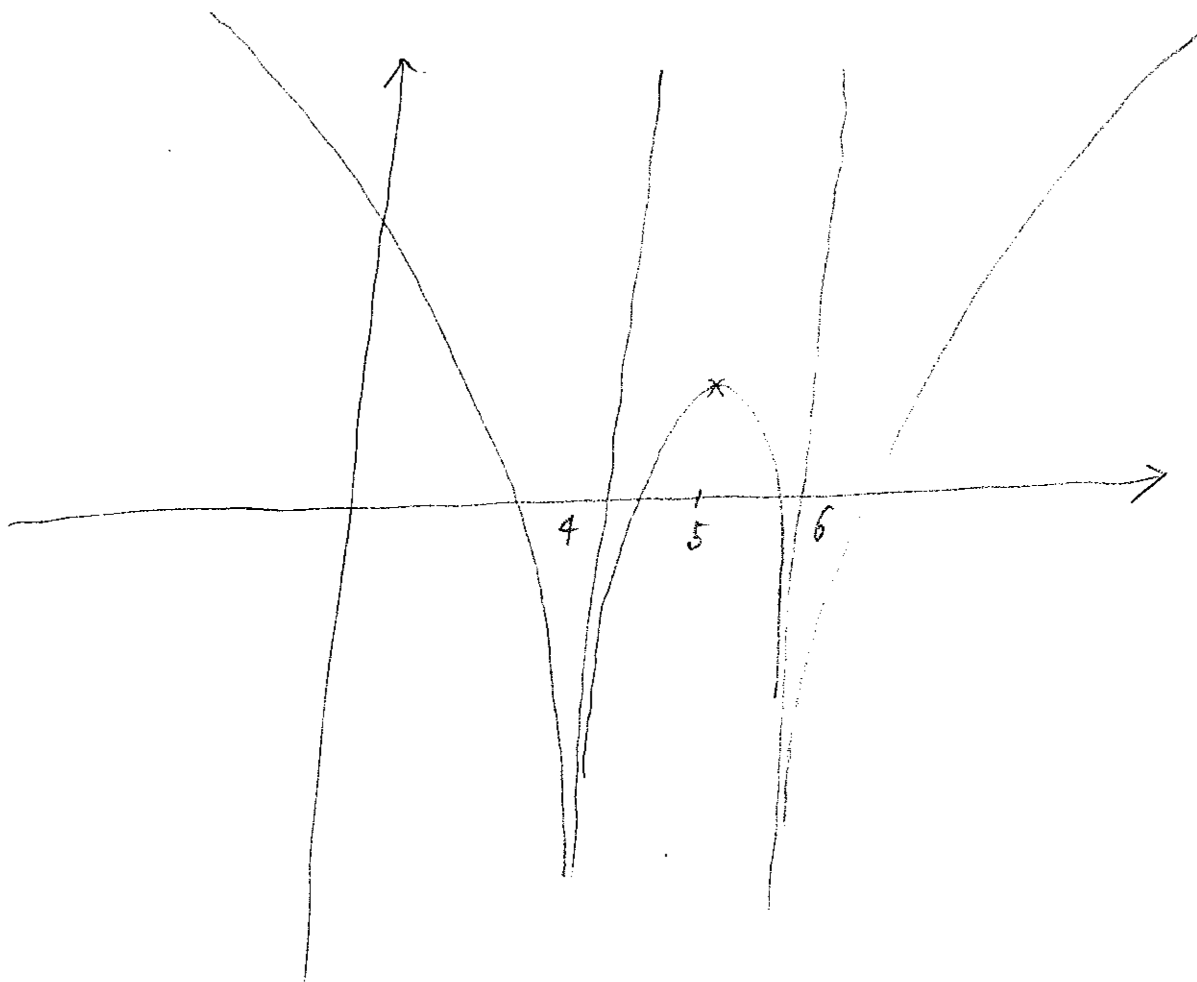
$$= \frac{-2\{(x-5)^2+1\}}{(x^2-10x+24)^2}$$

| $x$ | | 4 | | 5 | | 6 | |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | X | $+$ | 0 | $-$ | X | $+$ |
| $f''(x)$ | $-$ | X | $-$ | $-$ | $-$ | X | $-$ |
| $f(x)$ | $\searrow$ | X | $\nearrow$ | $\ln 24$ | $\searrow$ | X | $\nearrow$ |

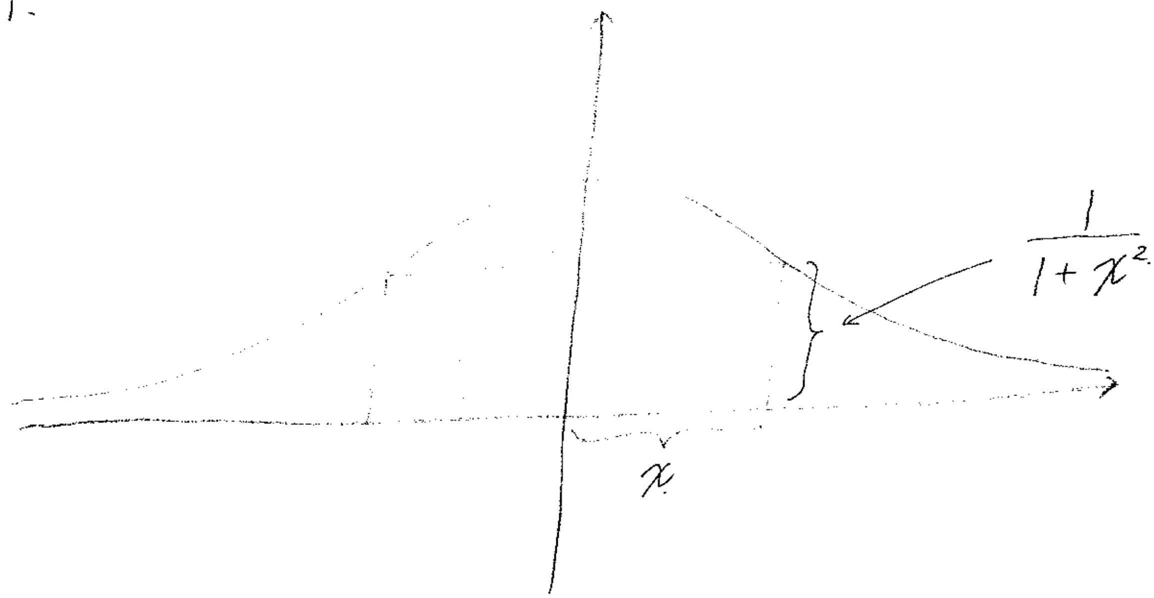
$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = \infty$$

$$\lim_{x\to 4^-} f(x) = -\infty \qquad \lim_{x\to 4^+} f(x) = -\infty$$

$$\lim_{x \to 6^-} f(x) = -\infty \qquad \lim_{x \to 6^+} f(x) = -\infty$$



24. 1.




$$A(x) = 2 \cdot x \cdot \frac{1}{1+x^2} = \frac{2x}{1+x^2}$$

Domain $x > 0$.

$$A'(x) = \frac{2\cdot(1+x^2) - 2x \cdot 2x}{(1+x^2)^2}$$

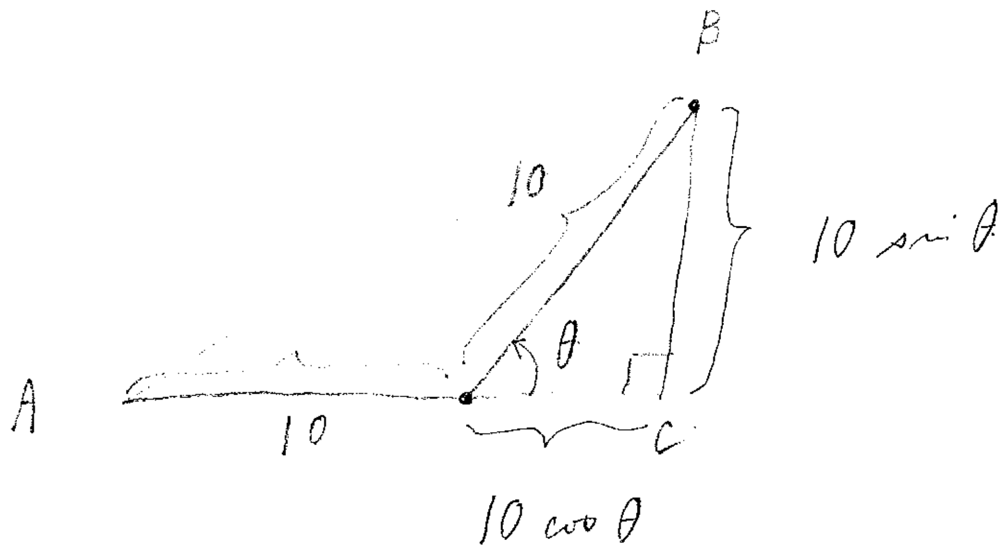
$$= \frac{-2(x^2-1)}{(1+x^2)^2}$$

| $x$ | 0 | | 1 | |
|---|---|---|---|---|
| $A'(x)$ | $\times$ | + | 0 | − |
| $A(x)$ | $\times$ | $\nearrow$ | 1 | $\searrow$ |

largest area.

$$A(1) = 2 \cdot 1 \cdot \frac{1}{1+1^2} = 1.$$

24.2.




$$A(\theta) = \frac{1}{2}(10 + 10\cos\theta)\cdot 10\sin\theta$$

$$= \frac{10^2}{2}(1+\cos\theta)\sin\theta$$

$$= 50(\sin\theta + \cos\theta\sin\theta)$$

$$A'(\theta) = 50(\cos\theta - \sin\theta\sin\theta + \cos\theta\cos\theta)$$

$$= 50\{\cos\theta - (1-\cos^2\theta) + \cos^2\theta\}$$

$$= 50(2\cos^2\theta + \cos\theta - 1)$$

$$= 50(2\cos\theta - 1)(\cos\theta + 1)$$

| $\theta$ | $0$ | | $\pi/3$ | | $\pi/2$ |
|---|---|---|---|---|---|
| $A'(\theta)$ | | $+$ | $0$ | $-$ | |
| $A(\theta)$ | | $\nearrow$ | | $\searrow$ | |

$$A\left(\frac{\pi}{3}\right) = 50\left\{\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right)\right\}$$

$$= 50\left\{\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right\}$$

$$= 50\cdot\frac{3\sqrt{3}}{4} = \frac{75\sqrt{3}}{2}$$

is the largest area.

25.1.

$$P(t) = P(0) \cdot 3^{\frac{t}{3}}$$
$$= P(0) \cdot e^{kt}$$

$\longrightarrow$

$$3^{\frac{t}{3}} = e^{kt}$$

$$\ln(\quad) = \ln(\quad)$$

$$\frac{t}{3} \ln 3 = k \cdot t.$$

$$\frac{\ln 3}{3} = k$$

( Time $t$ is measured in the unit of "weeks". )

$$700 = 50 \cdot 2^{\frac{t}{3}}$$

$$\frac{700}{50} = 2^{\frac{t}{3}}$$

$\parallel$

$14.$

$$\ln 14 = \frac{t}{3} \ln 2$$

$$\frac{3 \ln 14}{\ln 2} = t$$

Ans. $\frac{3 \ln 14}{\ln 2}$ hours

25. 2.

$$P(t) = P(0) \cdot e^{kt}$$
$$= 50 \cdot e^{kt}$$

$$P(3) = 50 \cdot e^{k \cdot 3}$$
$$\shortparallel$$
$$100$$

$\longrightarrow$

$$\frac{100}{50} = e^{3k}$$
$$\shortparallel$$
$$2$$

$\longrightarrow$

$$\ln 2 = 3k$$
$$\frac{\ln 2}{3} = k$$

$$\therefore \quad P(t) = 50 \cdot e^{\frac{\ln 2}{3} t}$$
$$= 50 \cdot (e^{\ln 2})^{\frac{t}{3}}$$
$$= 50 \cdot 2^{\frac{t}{3}}$$

25.3.

$$m(t) = m(0) \cdot 2^{-\frac{t}{30}}$$

$$= 60 \cdot 2^{-\frac{t}{30}}.$$

$$1 = 60 \cdot 2^{-\frac{t}{30}}.$$

$$\frac{1}{60} = 2^{-\frac{t}{30}}$$

$$\ln\left(\frac{1}{60}\right) = -\frac{t}{30} \ln 2$$

$$\shortparallel$$

$$-\ln 60$$

$$t = \frac{30 \cdot \ln 60}{\ln 2}$$

Ans. $\dfrac{30 \cdot \ln 60}{\ln 2}$ years.

25.4

$$m(t) = m(0) \cdot 2^{-\frac{t}{5730}}$$

$$\frac{74}{100} = \frac{m(t)}{m(0)} = 2^{-\frac{t}{5730}}$$

$$\frac{74}{100} = \frac{37}{50}$$

$$\ln\left(\frac{37}{50}\right) = -\frac{t}{5730} \cdot \ln 2$$

$$t = -5730 \cdot \frac{\ln\left(\frac{37}{50}\right)}{\ln 2}$$

years old.

26.1.

(i) $\int \tan x \, dx$

$$= \int \frac{\sin x}{\cos x} \, dx$$

$$\left( \begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array} \right)$$

$$= \int \frac{1}{u} (-du)$$

$$= -\int \frac{1}{u} \, du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C.$$

$$(= \ln|\sec x| + C)$$

26.1

(ii) $\displaystyle\int \frac{\ln x}{x}\,dx$ (99)

$$\left( \begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array} \right)$$

$$= \int u\,du = \frac{u^2}{2} + C$$

$$= \frac{(\ln x)^2}{2} + C$$

(iii) $\displaystyle\int_0^4 \sqrt{1+2x}\,dx$

$$\left( \begin{array}{lll} x & u = 1+2x & du = 2\,dx \\ 4 & 9 & \\ 0 & 1 & \end{array} \right)$$

$$= \int_1^9 \sqrt{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_1^9 \sqrt{u}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 = \frac{1}{2}\,\frac{2}{3}\left[3^3 - 1^3\right] = \frac{\cancel{52}\ 26}{3}$$

27

(iv) $\displaystyle\int_0^2 x^5\sqrt{1+x^2}\,dx$

$$\left(\begin{array}{lll} x & u = 1+x^2 & du = 2x\,dx \\ 2 & 5 & \\ 0 & 1 & \\ & x^2 = u-1 & \end{array}\right)$$

$$= \int_0^2 x^4\sqrt{1+x^2}\cdot x\,dx$$

$$= \int_1^5 (u-1)^2\sqrt{u}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_1^5 \left(u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du$$

$$= \frac{1}{2}\left[\frac{2}{7}u^{\frac{7}{2}} - 2\cdot\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right]_1^5$$

$$= \left[\left(\frac{1}{7}5^{\frac{7}{2}} - \frac{2}{5}5^{\frac{5}{2}} + \frac{1}{3}5^{\frac{3}{2}}\right) - \left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right)\right]$$

(v) $$\int_0^{\pi/4} \sec^4 x \tan x \, dx$$

$$\left( \begin{array}{lll} x & u = \tan x & du = \sec^2 x \, dx \\ \pi/4 & 1 & \\ 0 & 0 & \\ & \sec^2 x = 1 + u^2 & \end{array} \right)$$

$$= \int_0^{\pi/4} \sec^2 x \cdot \tan x \cdot \sec^2 x \, dx$$

$$= \int_0^1 (1 + u^2) u \, du = \int_0^1 (u^3 + u) \, du$$

$$= \left[ \frac{u^4}{4} + \frac{u^2}{2} \right]_0^1 = \frac{3}{4}$$

(vi) $$\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\, dx$$

$$\left( \begin{array}{ccc} x & u = \sin^{-1} x & du = \frac{1}{\sqrt{1-x^2}}\, dx \\ \frac{1}{2} & \frac{\pi}{6} & \\ 0 & 0 & \end{array} \right)$$

$$= \int_0^{\pi/6} u\, du = \left[ \frac{u^2}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{2} \left( \frac{\pi}{6} \right)^2 = \frac{\pi^2}{72}$$

(vii) $$\int_{\pi/6}^{\pi/3} \tan x \, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sin x}{\cos x} \, dx$$

$$\left( \begin{array}{lll} x & u = \cos x & du = -\sin x \, dx \\ \pi/3 & 1/2 & \\ \pi/6 & \sqrt{3}/2 & \end{array} \right)$$

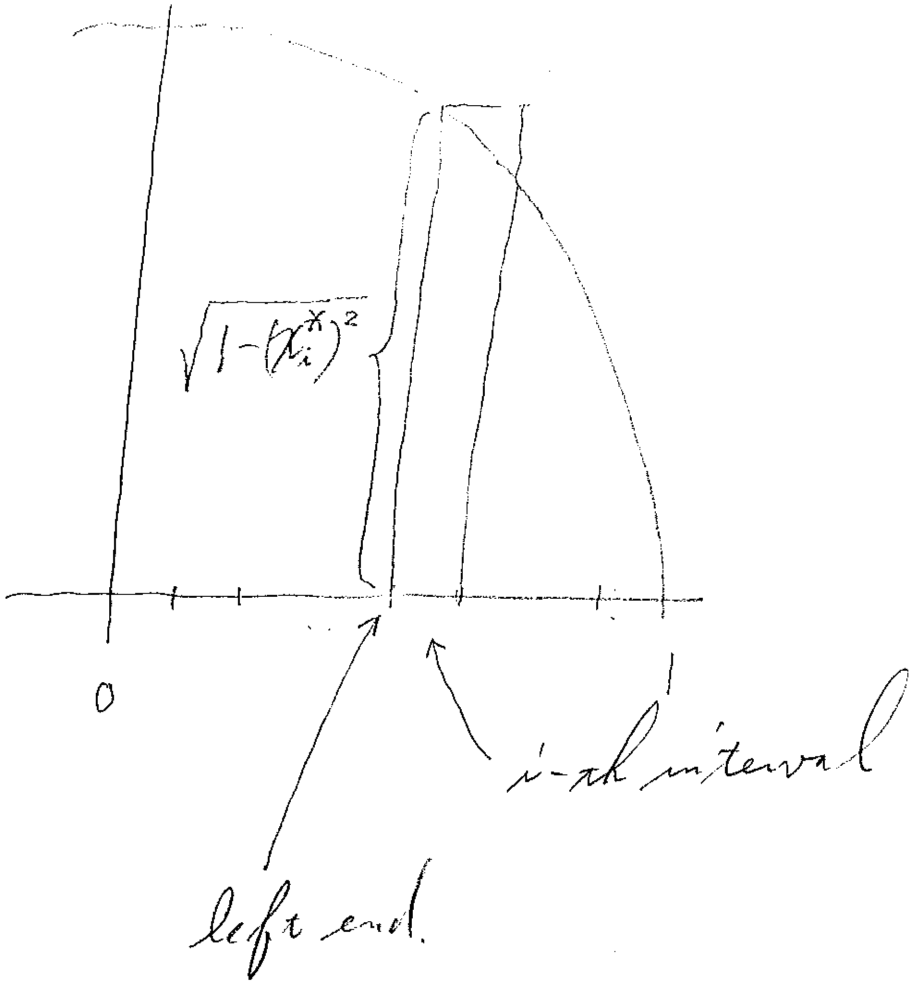
$$= \int_{\sqrt{3}/2}^{1/2} \frac{1}{u} (-du) = -\int_{\sqrt{3}/2}^{1/2} \frac{1}{u} \, du$$

$$= \int_{1/2}^{\sqrt{3}/2} \frac{1}{u} \, du = [\ln u]_{1/2}^{\sqrt{3}/2}$$

$$= \ln\left(\frac{\sqrt{3}}{2}\right) - \ln\left(\frac{1}{2}\right) = \left(\frac{1}{2}\ln 3 - \ln 2\right) - (-\ln 2)$$

$$= \frac{\ln 3}{2}.$$

27.1.




$x_i^*$ : $x$ - coordinate of the left end point of the i-th interval

$$= \frac{1-0}{n}\cdot(i-1) = \frac{i-1}{n}$$

$$\Delta x = \frac{1-0}{n} = \frac{1}{n}$$

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1-\left(\frac{i-1}{n}\right)^2} \cdot \frac{1}{n}.$$

27. 2

(i) $\lim_{n \to \infty} \sum_{i=1}^{n} \left( \sqrt{\underbrace{3 + i \cdot \frac{5}{n}}} \right) \cdot \frac{5}{n}$

$= \int_3^{3+5} \sqrt{x}\, dx$

$x_i^*$ : the $x$-coordinate of the right end point of the $i$-th interval obtained by subdividing $[3, 3+5]$ into $n$-subintervals

$\Delta x = \frac{(3+5) - 3}{n}$

$= \left[ \frac{2}{3} x^{\frac{3}{2}} \right]_3^8 = \frac{2}{3} \left[ 8^{\frac{3}{2}} - 3^{\frac{3}{2}} \right]$

$= \frac{2}{3} \left[ (2\sqrt{2})^3 - (\sqrt{3})^3 \right] = \frac{2}{3} \left[ 16\sqrt{2} - 3\sqrt{3} \right]$

(ii) $$\lim_{n\to\infty} \sum_{i=1}^{3n} \Big( \underbrace{1 + i \cdot \frac{2}{3n}}_{x_i^*} \Big)^5 \cdot \underbrace{\frac{2}{3n}}_{\Delta x}$$

$$= \int_1^{1+2} x^5 \, dx$$

$x_i^*$ : the $x$-coordinate of the right end point of the $i$-th interval obtained by subdividing $[1, 1+2]$ into $3n$-subintervals

$$\Delta x = \frac{(1+2) - 1}{3n}.$$

$$= \left[ \frac{x^6}{6} \right]_1^3 = \frac{1}{6} \left[ 3^6 - 1^6 \right] = \frac{728}{6}$$

$$= \frac{364}{3}.$$

29 (1)

(i) $y^2 + 4y + 4x + 8 = 0$

$$y^2 + 4y = -4x - 8$$

$$y^2 + 4y + 4 = -4x - 8 + 4$$

$$(y+2)^2 = -4(x+1)$$

$$\begin{cases} X = x + 1 \\ Y = y + 2 \end{cases} \qquad \begin{cases} x = X - 1 \\ y = Y - 2 \end{cases}$$

$$Y^2 = -4X$$

standard form

$$Y^2 = 4pX = 4(-1)X$$

XY world

focus $(p, 0) = (-1, 0)$

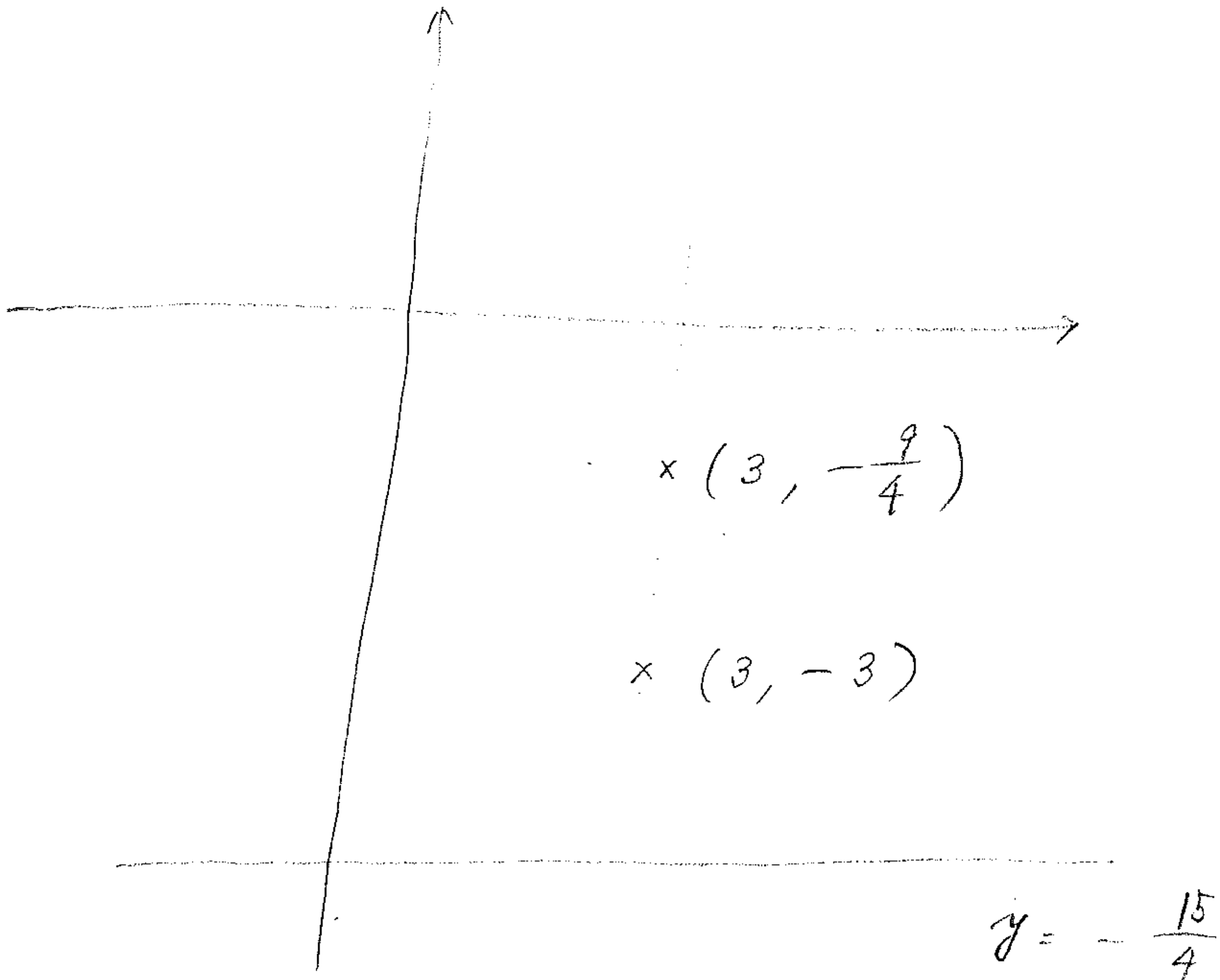
directrix $X = -p$ : $X = -(-1) = 1$

xy-world

focus $(-1-1,\ 0-2) = (-2, -2)$

$x + 1 = 1$ i.e. $x = 0$

(ii)




$xy$ - world

vertex $(3, -3)$

directrix $y = -\frac{15}{4}$.

$XY$ - world

vertex $(0, 0)$

$$\begin{cases} X = x - 3 \\ Y = y + 3 \end{cases} \qquad \begin{cases} x = X + 3 \\ y = Y - 3 \end{cases}$$

directrix

$$Y - 3 = -\frac{15}{4}$$

i.e.

$$Y = -\frac{3}{4} = -p$$

standard form.

$$X^2 = 4pY = 4 \cdot \frac{3}{4} \cdot Y$$

i.e.

$$X^2 = 3Y.$$

Going back to $xy$-world.

$$(x-3)^2 = 3(y+3)$$

29 (2)

(i) $16x^2 - 32x + 4y^2 + 4y = 47$

$16(x^2 - 2x \quad )$

$+ 4(y^2 + y \quad ) = 47$

$16(x^2 - 2x + 1)$

$+ 4(y^2 + y + \frac{1}{4}) = 47 + 16 + 1$

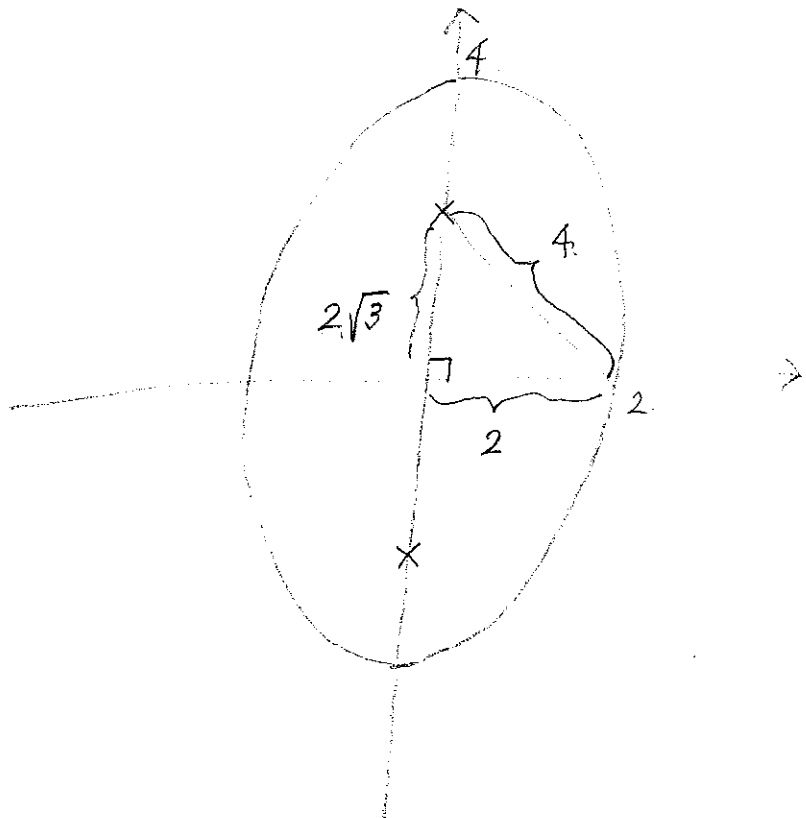
$16(x-1)^2 + 4(y + \frac{1}{2})^2 = 64$

$$\frac{(x-1)^2}{4} + \frac{(y+\frac{1}{2})^2}{16} = 1$$

$$\begin{cases} X = x - 1 \\ Y = y + \frac{1}{2} \end{cases} \qquad \begin{cases} x = X + 1 \\ y = Y - \frac{1}{2} \end{cases}$$

XY - word

$$\frac{X^2}{2^2} + \frac{Y^2}{4^2} = 1$$




standard from

$$\frac{X^2}{b^2} + \frac{Y^2}{a^2} = 1 \qquad (a \geq b)$$

$a = 4$ , $b = 2$

vertices $(0, \pm 4)$

foci $(0, \pm 2\sqrt{3})$

$|PF_1 + PF_2| = 2a = 2 \cdot 4 = 8$

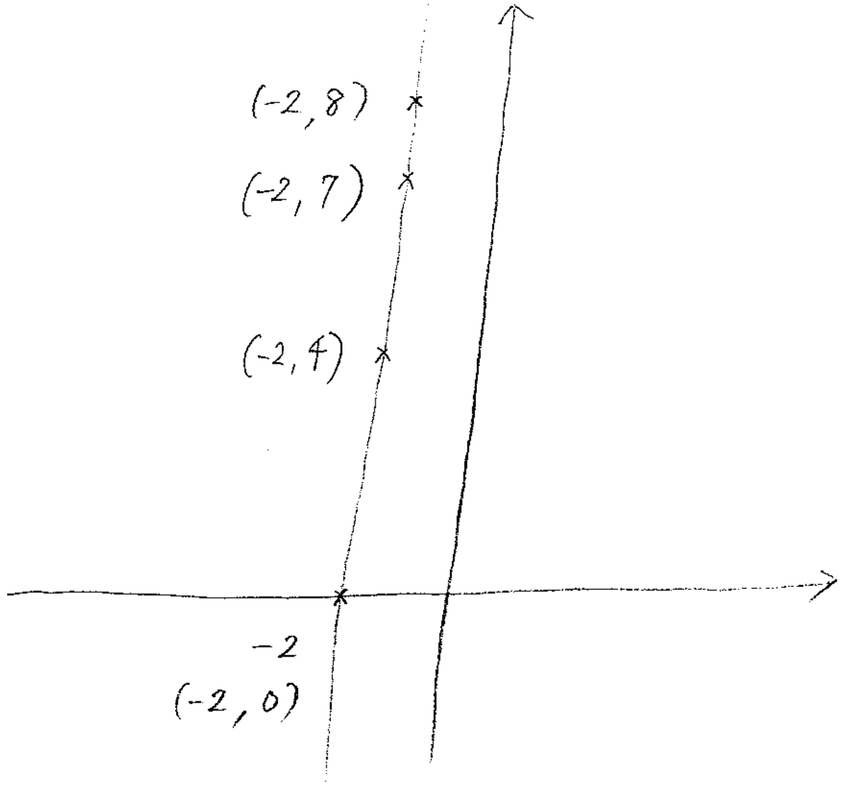
$xy$ - world

vertices $\left(0+1, \pm 4 - \frac{1}{2}\right)$

foci $\left(0+1, \pm 2\sqrt{3} - \frac{1}{2}\right)$

$$|PF_1 + PF_2| = 8$$

(ii)




xy - world

center $(-2, 4)$

XY - world

center $(0, 0)$

$$\begin{cases} X = x + 2 \\ Y = y - 4 \end{cases} \qquad \begin{cases} x = X - 2 \\ y = Y + 4 \end{cases}$$

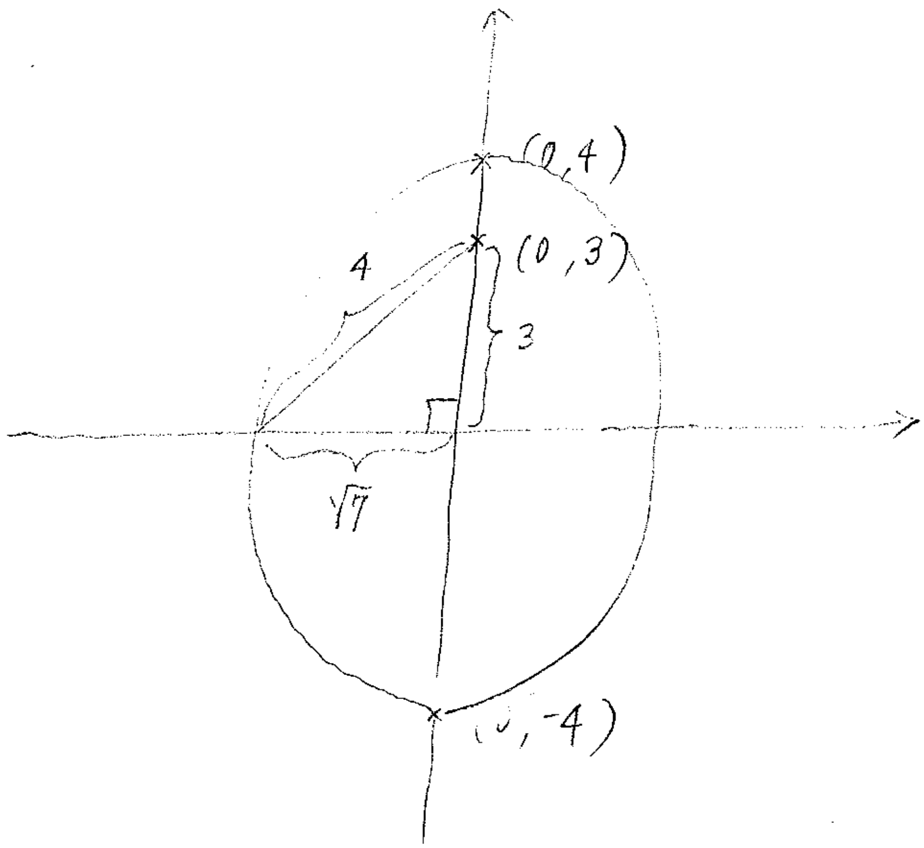
vertices

$(-2+2, 8-4) = (0, 4)$

$(-2+2, 0-4) = (0, -4)$

focus

$$(-2+2,\ 7-4) = (0, 3)$$




$$\frac{X^2}{(\sqrt{7})^2} + \frac{Y^2}{4^2} = 1.$$

xy - world

$$\frac{(x+2)^2}{(\sqrt{7})^2} + \frac{(y-4)^2}{4^2} = 1.$$

29 (3)

(i) $4x^2 - y^2 - 24x - 6y + 43 = 0$

$$4(x^2 - 6x \qquad)$$

$$-(y^2 + 6y \qquad) = -43$$

$$4(x^2 - 6x + 9)$$

$$-(y^2 + 6y + 9) = -43 + 36 - 9$$

$$4(x-3)^2 - (y+3)^2 = -16$$

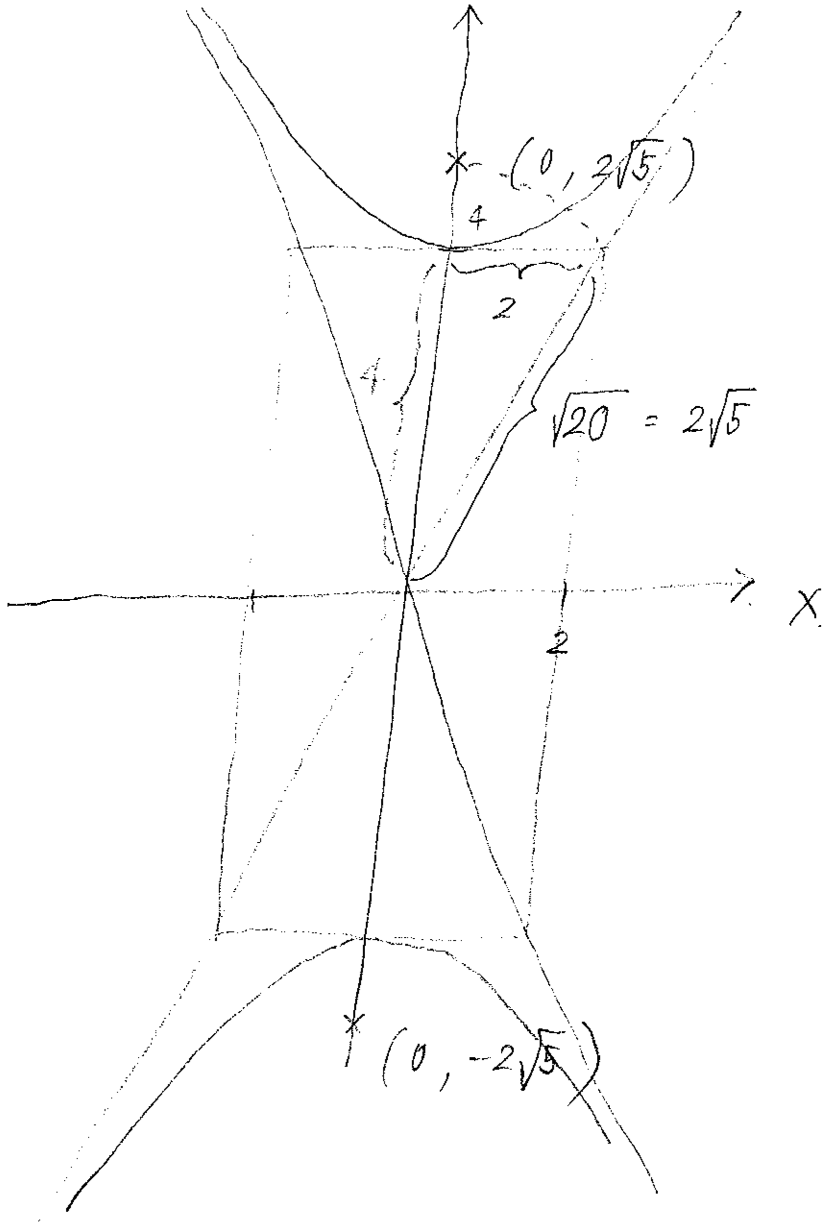
$$\frac{(y+3)^2}{16} - \frac{(x-3)^2}{4} = 1.$$

$$\begin{cases} X = x - 3 \\ Y = y + 3 \end{cases} \qquad \begin{cases} x = X + 3 \\ y = Y - 3 \end{cases}$$

XY - world

$$\frac{Y^2}{4^2} - \frac{X^2}{2^2} = 1$$

standard form $\dfrac{Y^2}{a^2} - \dfrac{X^2}{b^2} = 1.$




vertices $(0, \pm 4)$

foci $(0, \pm 2\sqrt{5})$

asymptotes $Y = \pm \frac{4}{2} X$

$|PF_1 - PF_2| = \pm 2a = \pm 2 \cdot 4$

xy - world

vertices $(0+3, \pm 4-3)$

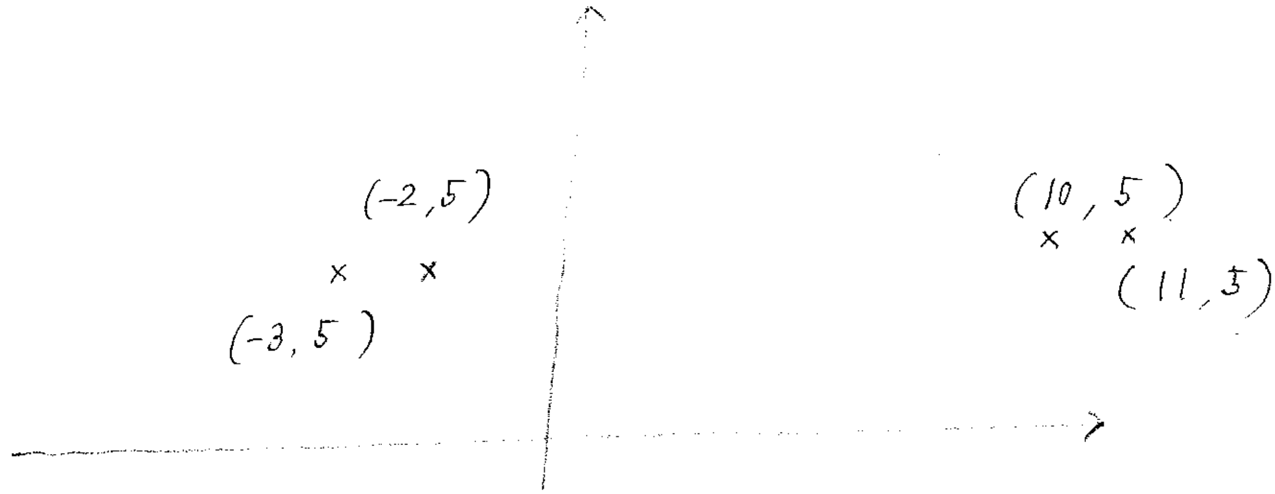
foci $(0+3, \pm 2\sqrt{5}-3)$

asymptotes

$y+3 = \pm 2(x-3)$

$|PF_1 - PF_2| = \pm 8$

(ii)




$xy$-world

vertices $(-2, 5)$ $(10, 5)$

foci $(-3, 5)$ $(11, 5)$

center $(4, 5)$

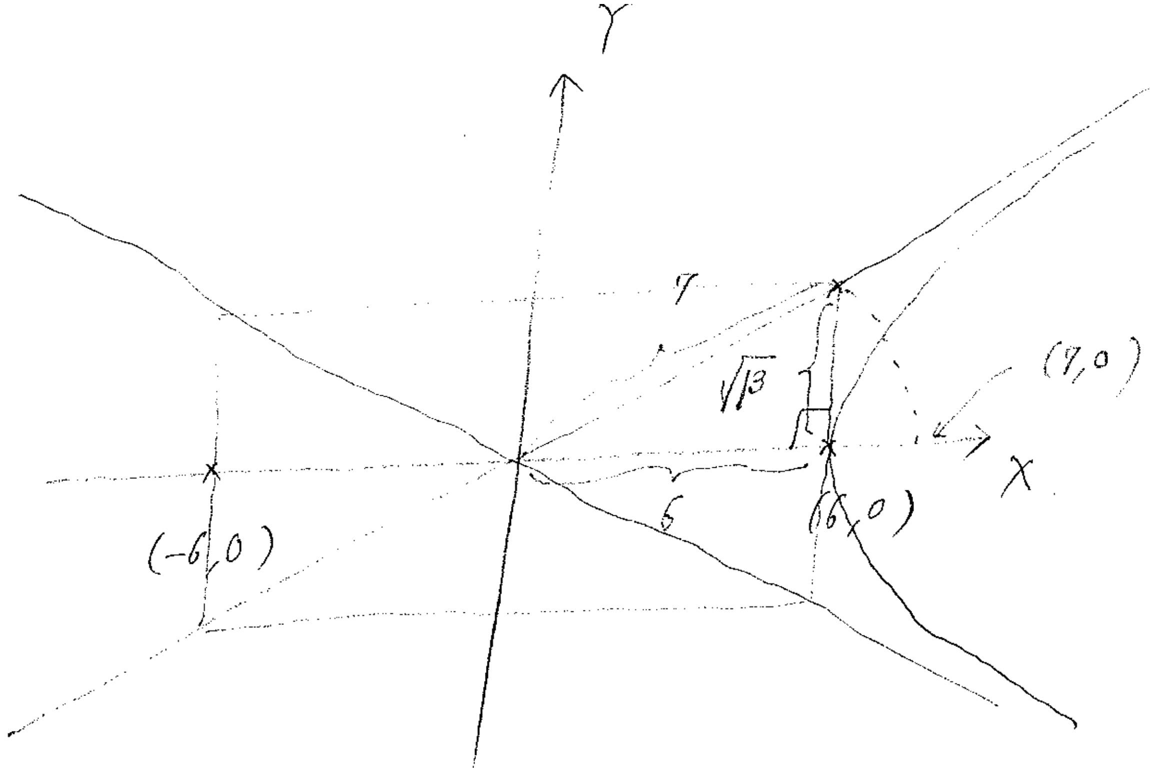
$XY$-world

center $(0, 0)$

$$\begin{cases} X = x - 4 \\ Y = y - 5 \end{cases} \qquad \begin{cases} x = X + 4 \\ y = Y + 5 \end{cases}$$

vertices $(-6, 0)$ $(6, 0)$

foci $(-7, 0)$ $(7, 0)$




$$\frac{X^2}{6^2} - \frac{Y^2}{(\sqrt{13})^2} = 1$$

xy - wald

$$\frac{(x-4)^2}{6^2} - \frac{(y-5)^2}{(\sqrt{13})^2} = 1$$