Answer Keys for Study Guide for Exam 3

1.1.

Step 1.

$$f(x) = 3x^4 - 16x^3 + 18x^2$$

on $[-1, 4]$.

Step 2.

① $a = -1$, $b = 4$.

② $$f'(x) = 12x^3 - 48x^2 + 36x$$
$$= 12x(x^2 - 4x + 3)$$
$$= 12x(x-1)(x-3)$$

$f'(c) = 0 \quad c = 0, 1, 3$

Step 3.

$f(-1) = 37 \quad \leftarrow$ abs. max.

$f(4) = 32$

$f(0) = 0$

$f(1) = 3$

$f(3) = -27 \quad \leftarrow$ abs. mini.

1. 2.

$16 \quad -12 \quad -24 \quad -1$

(a) Step 1

$-16 \quad -12 \quad +24 \quad +1$

$54 \quad -27 \quad -36 \quad +1$

$$f(x) = 2x^3 - 3x^2 - 12x + 1$$

$-2 \quad -3 \quad +12$

on $[-2, 3]$.

Step 2.

① $a = -2, \quad b = 3$

② $f'(x) = 6x^2 - 6x - 12$

$= 6(x^2 - x - 2)$

$= 6(x+1)(x-2)$

$f'(c) = 0 \qquad c = -1, 2.$

Step 3

$f(-2) = -3$

$f(3) = -8$

$f(-1) = 8 \quad \longleftarrow$ abs. max

$f(2) = -19 \quad \longleftarrow$ abs. min

(b) Step 1.

$$f(x) = xe^{\frac{x}{2}} \quad \text{on} \quad [-3, 1]$$

Step 2

① $a = -3$, $b = 1$.

② $f'(x) = 1 \cdot e^{\frac{x}{2}} + x \cdot \frac{1}{2} e^{\frac{x}{2}}$

$$= \left(1 + \frac{x}{2}\right) e^{\frac{x}{2}}$$

$f'(c) = 0 \qquad c = -2.$

Step 3 $\quad f(-3) = -3e^{-\frac{3}{2}}$

$f(1) = 1 \cdot e^{\frac{1}{2}} \leftarrow$ abs. max.

$f(-2) = -2e^{-\frac{2}{2}} \leftarrow$ abs. min.

(C) Step 1.

$f(x) = (x^2 - 1)^3$ on $[-1, 3]$

Step 2

① $a = -1$, $b = 3$

② $f'(x) = 3(x^2-1) \cdot 2x$

$f'(c) = 0$.

$c = \pm 1, 0$

Step 3

$f(-1) = 0$

$f(3) = 8$ ← abs. max.

$f(1) = 0$.

$f(0) = -1$ ← abs. min.

(d) Step 1.

$$f(t) = 2\cos t + \sin 2t \quad \text{on} \quad [0, 2\pi]$$

Step 2

① $a = 0$, $b = 2\pi$

② $f'(t) = -2\sin t + 2\cos 2t$

$= -2\sin t + 2(1 - 2\sin^2 t)$

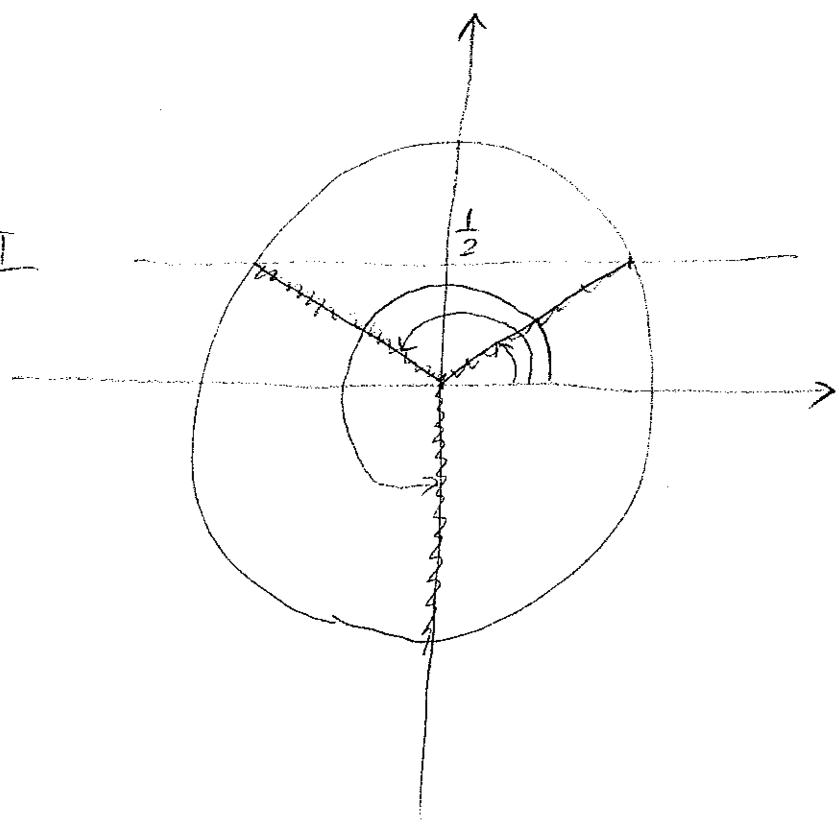
$= -2(2\sin^2 t + \sin t - 1)$

$= -2(2\sin t - 1)(\sin t + 1)$

$f'(c) = 0$

$c = \frac{\pi}{6}, \frac{5\pi}{6}$

$\frac{3\pi}{2}$.



Step 3.

$$f(0) = 2.$$

$$f(2\pi) = 2$$

abs. max

$$f\left(\frac{\pi}{6}\right) = 2 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

$$f\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$$

abs. min.

$$f\left(\frac{3\pi}{2}\right) = 0$$

(e)

Step 1.

$f(x) = \ln(x^2 + x + 1)$ on $[-1, 1]$

$\frac{1}{4} \quad -\frac{1}{2}$

Step 2.

① $a = -1, \quad b = 1.$

② $f'(x) = \dfrac{2x + 1.}{x^2 + x + 1.}$

$f'(c) = 0 \qquad c = -\dfrac{1}{2}$

Step 3.

$f(-1) = 0.$

$f(1) = \ln 3 > 0 \leftarrow$ abs. max

$f(-\frac{1}{2}) = \ln(\frac{3}{4}) < 0 \leftarrow$ abs. min

(f)

Step 1.

$$f(x) = x - \sqrt{x} \quad \text{on } [0, 9]$$

Step 2

① $a = 0, \quad b = 9$

② $f'(x) = 1 - \dfrac{1}{2\sqrt{x}}$

$f'(c) = 0 \qquad c = \dfrac{1}{4}$

Step 3

$f(0) = 0$

$f(3) = 6 \qquad \leftarrow$ abs. max

$f\left(\dfrac{1}{4}\right) = -\dfrac{1}{4} \qquad \leftarrow$ abs. min.

2.1

$$f'(x) = (x+2)^2(x+1)(x-1)^3(x-3)^2(x-5)$$

x		-2		-1		1		3		5	
$f'(x)$	$-$	0	$-$	0	$+$	0	$-$	0	$-$	0	$+$
$f(x)$	$\searrow$		$\searrow$		$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

(a) local max $x = 1$.

(b) local min $x = -1, 5$

2.2.

(a) $f(x) = x^8 (x-4)^7$

$$f'(x) = 8x^7 (x-4)^7 + x^8 \cdot 7(x-4)^6$$

$$= x^7 (x-4)^6 \{ 8(x-4) + x \cdot 7 \}$$

$$= x^7 (x-4)^6 (15x - 32)$$

critical numbers

$$0,\ 4,\ \frac{32}{15}$$

(b) $f''(x)$

$$= 7x^6 \cdot (x-4)^6 (15x-32) + x^7 \cdot 6(x-4)^5 (15x-32) + x^7 \cdot (x-4)^6 \cdot 15$$

$$= x^6 (x-4)^5 \cdot \{ 7 \cdot (x-4)(15x-32) + x (15x-32) + x \cdot (x-4) \cdot 15 \}$$

$x = 0$ $\quad f''(0) = 0$

$\longrightarrow$ 2nd Der. Test inconclusive

$x = 4$ $\quad f''(4) = 0$

$\longrightarrow$ 2nd Der. Test inconclusive

$x = \frac{32}{15}$ $\quad f''\left(\frac{32}{15}\right)$

$$= \underbrace{\left(\frac{32}{15}\right)^6}_{> 0} \underbrace{\left(\frac{32}{15} - 4\right)^5}_{< 0} \underbrace{\left\{\frac{32}{15} \cdot \left(\frac{32}{15} - 4\right) \cdot 15\right\}}_{< 0}$$

> 0

$\longrightarrow$ 2nd Der. Test

local min

(c)

x		0		$\frac{32}{15}$		4	
$f'(x)$	$+$	0	$-$	0	$+$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$		$\nearrow$

$x = 0$ local max

$x = \frac{32}{15}$ local min.

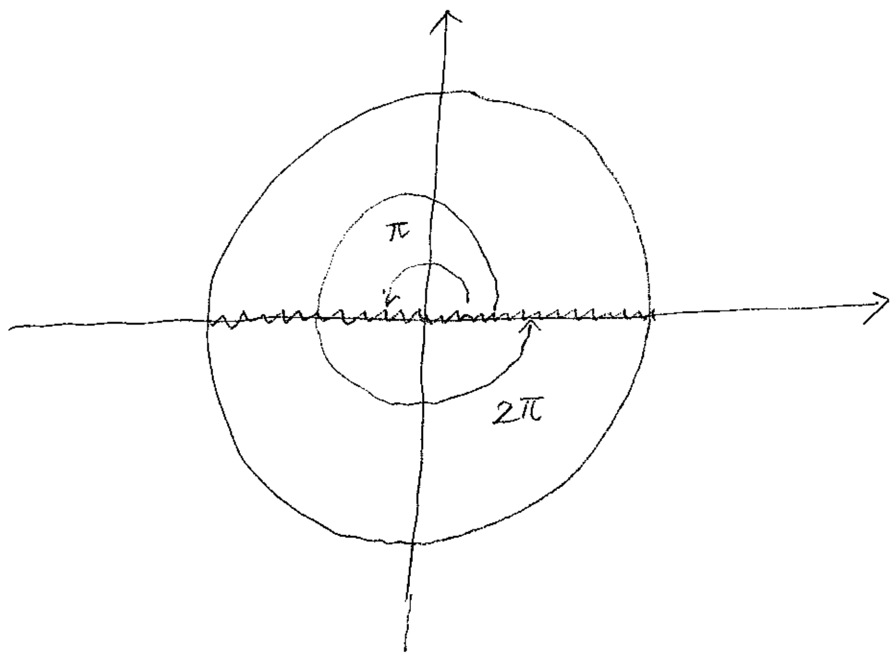
$x = 4$ neither local max nor local min.

3.1

$$f(x) = \frac{1}{2}x - \sin x$$

$$f'(x) = \frac{1}{2} - \cos x$$

$$f''(x) = \sin x$$



x	0		π		2π		3π
$f''(x)$	X	+	0	−	0	+	
	X	concave up		concave down		concave up.	

3.2.

$$f''(x) = (x+5)^3 (x+2)^2 (x-2)^5 (x-3)^3 (x-6)^2$$

x		-5		-2		2		3		6	
$f''(x)$	$-$	0	$+$	0	$+$	0	$-$	0	$+$	0	$+$
		infl. pt.				infl pt		infl pt			

$x = -5, \ 2, \ 3.$

These are NOT inflection points.

3.3.

$$f(x) = x^5 - 5x^4 + 25x$$

$$f'(x) = 5x^4 - 20x^3 + 25$$

$$f''(x) = 20x^3 - 60x^2$$

$$= 20x^2(x-3)$$

x		0		3	
$f''(x)$	$-$	0	$-$	0	$+$

NOT infl pt. (at $x = 0$)

infl pt. (at $x = 3$)

Ans. Only 1 inflection pt.

4.

(a) $\lim_{x\to\infty} \dfrac{\ln(x)}{\sqrt{x}}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{1/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$

(b) $\lim_{x\to 0} \dfrac{1-\cos x}{3x^2}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\sin x}{6x}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}$

(c) $\lim_{x\to 0} \dfrac{\sin x}{1-x^2} = 0$

Warning : Do NOT use L'Hospital's rule!

(d) $\lim_{x\to 0} \dfrac{3x - \sin(3x)}{3x - \tan(3x)}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{3 - 3\cos(3x)}{3 - 3\sec^2(3x)}$ $\quad \left(\dfrac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{3\cdot 3\sin(3x)}{-3\cdot 2\sec(3x)\cdot \sec(3x)\tan(3x)\cdot 3}$

$$= \lim_{x \to 0} \frac{9\,\cancel{\sin(3x)}}{-18\,\underbrace{\sec^2(3x)}_{1}\,\frac{\cancel{\sin(3x)}}{\underbrace{\cos(3x)}_{1}}} = -\frac{1}{2}.$$

(e) $$\lim_{x \to 0} \frac{\sin x - x}{x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x}{6} = -\frac{1}{6}$$

5.

(a) $\displaystyle \lim_{x\to 0} \underbrace{\sin x}_{\to\, 0}\ \underbrace{\ln(2x)}_{\to\,(-\infty)} \qquad (0 \times (-\infty))$

$$= \lim_{x\to 0} \frac{\ln(2x)}{1/\sin x}$$

$$= \lim_{x\to 0} \frac{\frac{\cancel{2}}{\cancel{2}x}}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0} -\frac{\sin^2 x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} -\frac{2\underbrace{\sin x}_{0}\cdot\underbrace{\cos x}_{1}}{1} = 0.$$

(b) $\displaystyle \lim_{x\to\infty} \underbrace{2x}_{\to\,\infty}\ \underbrace{\tan\left(\frac{1}{3x}\right)}_{\to\,0} \qquad (\infty \times 0)$

$$= \lim_{x\to\infty} \frac{\tan\left(\frac{1}{3x}\right)}{\frac{1}{2x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{3x}\right)\cdot\frac{1}{3}\left(-\cancel{\frac{1}{x^2}}\right)}{\frac{1}{2}\left(-\cancel{\frac{1}{x^2}}\right)} = \frac{2}{3}$$

(c) $$\lim_{x \to \left(\frac{\pi}{2}\right)^-} \overbrace{(2x - \pi)}^{0} \, \overbrace{\tan x}^{+\infty} \qquad (0 \times \infty)$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\tan x}{1/(2x-\pi)} \qquad \left(\frac{\infty}{-\infty}\right)$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{\sec^2 x}{-\dfrac{2}{(2x-\pi)^2}}$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} -\frac{(2x-\pi)^2}{2\cos^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \cancel{-} \frac{\cancel{2}(2x-\pi)\cdot\cancel{2}}{\cancel{2}\cdot 2\cos x\,(\cancel{-}\sin x)}$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{2x-\pi}{\cos x \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{2}{-\sin x \sin x + \cos x \cos x} = -2$$

(d) $$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$$

$$\left(\lim_{x\to 1} \quad \underbrace{}_{\text{DNE}} \quad \underbrace{}_{\text{DNE}} \right)$$

$$= \lim_{x\to 1} \frac{x \ln x - (x-1)}{(x-1)\ln x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{1 \cdot \ln x + x \cdot \frac{1}{x} - 1}{1 \cdot \ln x + (x-1)\frac{1}{x}}$$

$$= \lim_{x\to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1} \frac{1/x}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}.$$

(e) $$\lim_{x \to 4} \left(\underbrace{\frac{1}{\sqrt{x} - 2}} - \underbrace{\frac{4}{x - 4}} \right)$$

$$\lim_{x \to 4} \quad \text{DNE} \qquad \text{DNE}$$

$$= \lim_{x \to 4} \frac{(\sqrt{x} + 2) - 4}{x - 4}$$

$$= \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$$

$$= \lim_{x \to 4} \frac{(\sqrt{x} - 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)} = \frac{1}{4}$$

6. (a) $\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{7x}$ $\qquad (1^{\infty})$

Set $y = \left(1+\frac{3}{x}\right)^{7x}$

$$\ln y = 7x \cdot \ln\left(1+\frac{3}{x}\right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \underbrace{7x}_{\to\infty} \cdot \underbrace{\ln\left(1+\frac{3}{x}\right)}_{\to 0} \quad (\infty \times 0)$$

$$= \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{7x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}} \cdot 3 \cdot \cancel{\left(-\frac{1}{x^2}\right)}}{\frac{1}{7}\cdot\cancel{\left(-\frac{1}{x^2}\right)}} = 21.$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y \to 21} = e^{21}.$$

(b) $\lim_{x \to \infty} \left(\frac{2x+1}{2x-1} \right)^{4x+1}$ $\quad (1^\infty)$

Set

$$y = \left(\frac{2x+1}{2x-1} \right)^{4x+1}$$

$$\ln y = (4x+1) \ln \left(\frac{2x+1}{2x-1} \right)$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \underbrace{(4x+1)}_{\to\,\infty} \underbrace{\ln \Big(\underbrace{\frac{2x+1}{2x-1}}_{\to\,1} \Big)}_{\to\,0}$$

$$\ln(2x+1) - \ln(2x-1)$$

$$(\infty \times 0)$$

$$= \lim_{x\to\infty} \frac{\ln \left(\frac{2x+1}{2x-1} \right)}{1/(4x+1)} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{2}{2x+1} - \frac{2}{2x-1}}{-\frac{4}{(4x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{\dfrac{2\{\overbrace{(2x-1) - (2x+1)}^{-2}\}}{(2x+1)(2x-1)}}{-\dfrac{4}{(4x+1)^2}}$$

$$= \lim_{x \to \infty} \frac{(4x+1)^2}{(2x+1)(2x-1)} = 4.$$

i.e.

$$\lim_{x \to \infty} \ln y = 4.$$

$$\therefore \lim_{x \to \infty} y = \lim_{x \to \infty} e^{\overbrace{\ln y}^{\to 4}} = e^4.$$

(c) $\lim_{x\to\infty} (2x + e^{5x})^{\frac{1}{x}}$

Set $y = (2x + e^{5x})^{\frac{1}{x}}$

$$\ln y = \frac{1}{x} \ln (2x + e^{5x})$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \underbrace{\frac{1}{x}}_{\downarrow \atop 0} \underbrace{\ln (2x + e^{5x})}_{\downarrow \atop \infty}$$

$$(0 \times \infty)$$

$$= \lim_{x\to\infty} \frac{\ln (2x + e^{5x})}{x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2x + e^{5x}} (2 + 5e^{5x})}{1}$$

$$= \lim_{x\to\infty} \frac{2 + 5e^{5x}}{2x + e^{5x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to\infty} \frac{5 \cdot 5e^{5x}}{2 + 5e^{5x}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{5 \cdot \cancel{5} \cdot \cancel{5} \cancel{e^{5x}}}{\cancel{5} \cdot \cancel{5} \cdot \cancel{e^{5x}}} = 5$$

i.e.

$$\lim_{x \to \infty} \ln y = 5.$$

$$\therefore \lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln y \to 5} = e^{5}$$

(d) $\lim_{x \to 0^+} \tan(5x)^{\sin x}$

$$\text{Let } y = \tan(5x)^{\sin x}$$

$$\ln y = \sin x \ln \{\tan(5x)\}$$

$$\lim_{x\to 0^+} \ln y = \lim_{x \to 0^+} \underbrace{\sin x}_{\downarrow \atop 0} \underbrace{\ln \{\underbrace{\tan(5x)}_{\downarrow \atop 0}\}}_{\downarrow \atop -\infty}$$

$$(0 \times (-\infty))$$

$$= \lim_{x\to 0^+} \frac{\ln\{\tan(5x)\}}{\frac{1}{\sin x}} \left(\frac{-\infty}{+\infty}\right)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\tan(5x)} \cdot \sec^2(5x) \cdot 5}{-\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{\cos(5x)}{\sin(5x)} \cdot \sin^2 x \cdot 5}{-\cos x \cdot \cos^2(5x)}$$

$$= \lim_{x\to 0^+} \underbrace{\frac{\cos(5x)\cdot 5}{-\cos x \cdot \cos^2(5x)}}_{\to -5} \cdot \underbrace{\frac{\sin^2 x}{\sin(5x)}}_{\to 0}$$

$$\left(\begin{aligned} &\lim_{x\to 0^+} \frac{\sin^2 x}{\sin 5x} \quad \left(\frac{0}{0}\right) \\ &= \lim_{x\to 0^+} \frac{2\sin x \cos x}{5\cos 5x} = 0 \end{aligned} \right)$$

i.e.

$$\lim_{x\to 0^+} \ln y = 0$$

$$\therefore \lim_{x\to 0^+} y = \lim_{x\to 0^+} e^{\overbrace{\ln y}^{\to 0}} = e^0 = 1.$$

7.3.

Consider the function

$$f(x) = \sin^{-1} x + \cos^{-1} x$$

Then

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \left(-\frac{1}{\sqrt{1-x^2}}\right) = 0.$$

$\longrightarrow$ $f(x)$ constant

$\longrightarrow$ $f\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{1}{5}\right) + \cos^{-1}\left(\frac{1}{5}\right)$

$$\begin{aligned} &\parallel \\ f(0) &= \sin^{-1}(0) + \cos^{-1}(0) \\ &= 0 + \pi/2 = \pi/2 \end{aligned}$$

7.4.

Consider the function

$$f(x) = x^3 + x - 1.$$

f cont. on $[0, 1]$

$$f(0) = -1 < \underset{N.}{\overset{\shortparallel}{0}} < 1 = f(1)$$

$\xrightarrow[\text{I.M.V.}]{}$ $\exists\ c \in (0, 1)$ s.t. $f(c) = N = 0$.

That is to say, there exists at least one solution c on the interval $[0, 1]$.

We claim this c is the unique solution. Suppose there is another solution $c' \in [0, 1]$ $(c' \neq c)$ s.t. $f(c') = 0$.

(Say $c < c'$.)

Then by M.V.Th, we should have

$d \in (c, c')$

s.t. $f'(d) = \dfrac{f(c') - f(c)}{c' - c} = \dfrac{0-0}{c'-c} = 0.$

On the other hand,

$f'(d) = 3d^2 + 1 > 0.$

This is a contradiction!

That means there is no solution on $[0,1]$ other than c.

$\#$ of solution(s) on $[0,1]$.

$= 1.$

8.1.

(a) $$y = f(x) = \frac{x}{x^2-16}$$

Domain $x \neq \pm 4$

$$f'(x) = \frac{1\cdot(x^2-16) - x\cdot 2x}{(x^2-16)^2}$$

$$= \frac{-(x^2+16)}{(x^2-16)^2}$$

$$f''(x) = \frac{(-2x)(x^2-16)^2 + (x^2+16)2(x^2-16)\cdot 2x}{(x^2-16)^4}$$

$$= \frac{(-2x)(x^2-16) + (x^2+16)4x}{(x^2-16)^3}$$

$$= \frac{2x(x^2+48)}{(x^2-16)^3}$$

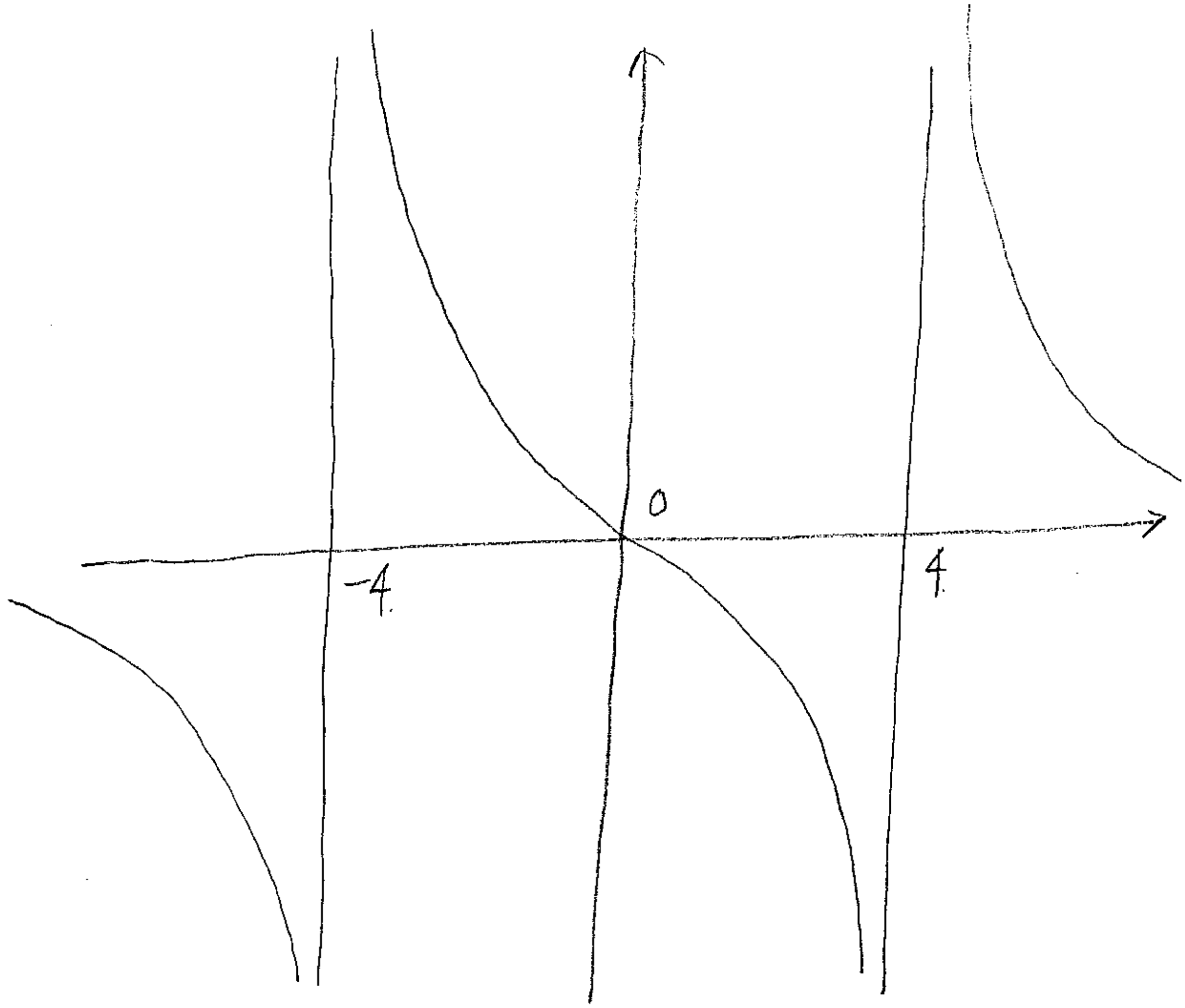
x		-4		0		4	
$f'(x)$	$-$	X	$-$	$-$	$-$	X	$-$
$f''(x)$	$-$	X	$+$	0	$-$	X	$+$
$f(x)$		X				X	

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{x}{x^2-16} = -\infty$$

$$\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} \frac{x}{x^2-16} = +\infty$$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} \frac{x}{x^2-16} = -\infty$$

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \frac{x}{x^2-16} = +\infty$$


0
-4
4

(b) $y = f(x) = \dfrac{x}{x^2+16}$

Domain $(-\infty, \infty)$

$$f'(x) = \frac{1\cdot(x^2+16) - x\cdot 2x}{(x^2+16)^2}$$

$$= \frac{-(x^2-16)}{(x^2+16)^2}$$

$$= \frac{-(x+4)(x-4)}{(x^2+16)^2}$$

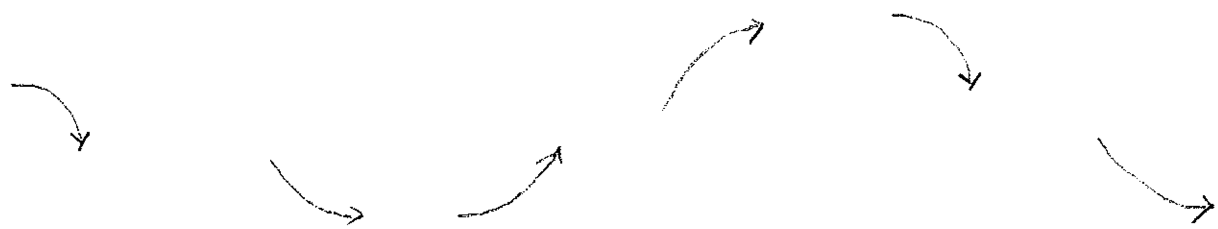
$$f''(x) = \frac{(-2x)(x^2+16)^2 + (x^2-16)2(x^2+16)\cdot 2x}{(x^2+16)^4}$$

$$= \frac{(-2x)(x^2+16) + (x^2-16)4x}{(x^2+16)^3}$$

$$= \frac{2x^3 - 96x}{(x^2+16)^3}$$

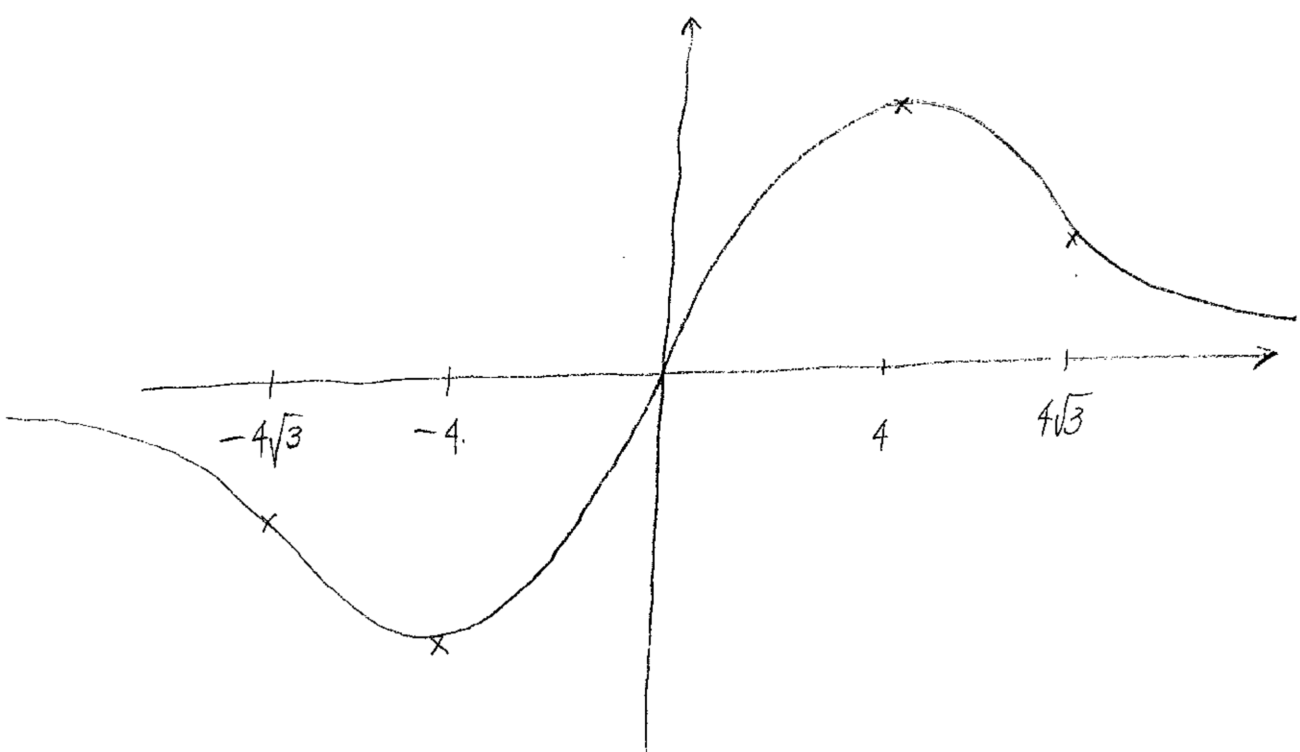
$$= \frac{2x(x+4\sqrt{3})(x-4\sqrt{3})}{(x^2+16)^3}$$

x		$-4\sqrt{3}$		-4		0		4		$4\sqrt{3}$	
$f'(x)$	$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$
$f''(x)$	$-$	0	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$
$f(x)$											



$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+16} = 0$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x^2+16} = 0$$



(c) $\quad y = f(x) = \dfrac{1}{x^2-16}$

Domain $\quad x \neq \pm 4$

$$f'(x) = \frac{-2x}{(x^2-16)^2}$$

$$f''(x) = \frac{-2(x^2-16)^2 + 2x \cdot 2(x^2-16)\cdot 2x}{(x^2-16)^4}$$

$$= \frac{-2(x^2-16) + 8x^2}{(x^2-16)^3}$$

$$= \frac{6x^2 + 32}{(x^2-16)^3}$$

x		-4		0		4	
$f'(x)$	$+$	$\times$	$+$	0	$-$	$\times$	$-$
$f''(x)$	$+$	$\times$	$-$	$-$	$-$	$\times$	$+$
$f(x)$		$\times$				$\times$	

$$\lim_{x\to\infty} \frac{1}{x^2-16} = 0$$

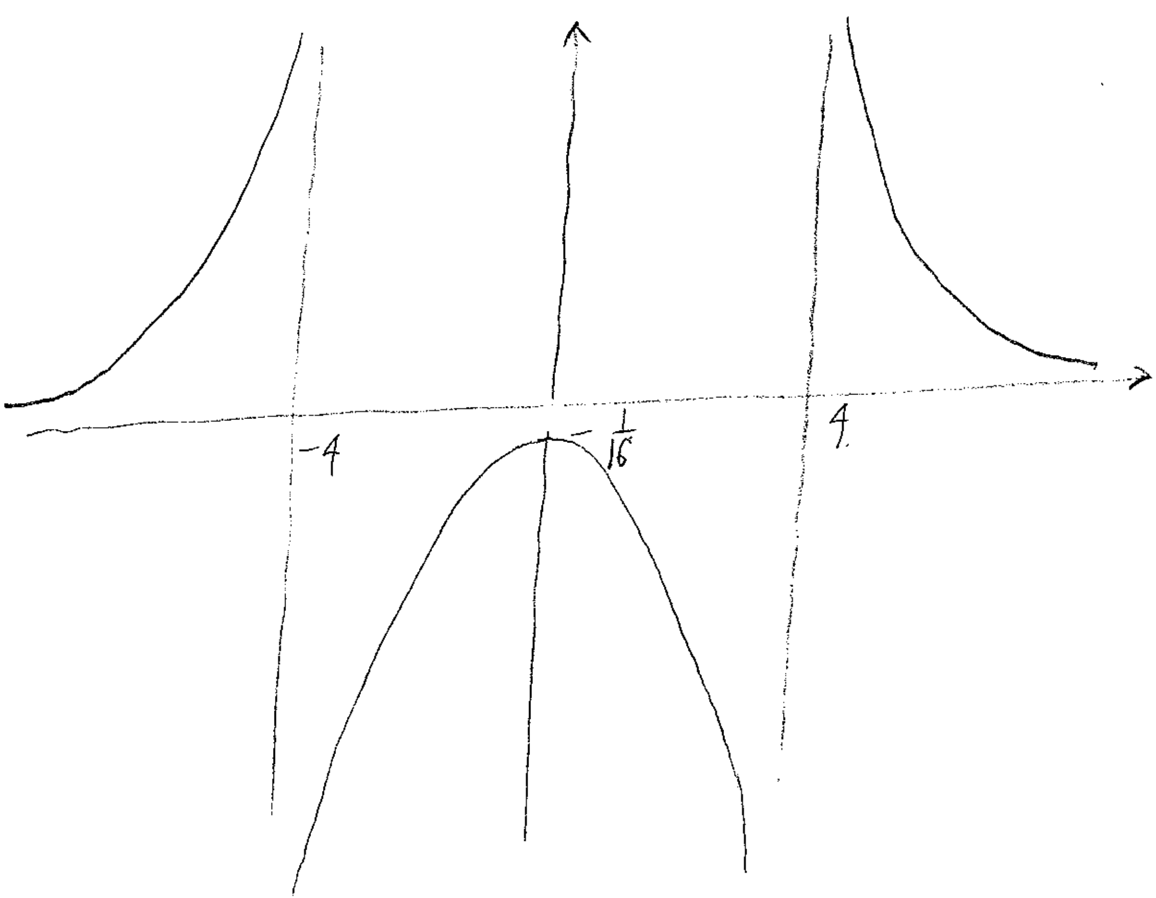
$$\lim_{x\to-\infty} \frac{1}{x^2-16} = 0$$

$$\lim_{x\to-4^-} \frac{1}{x^2-16} = +\infty$$

$$\lim_{x\to-4^+} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^-} \frac{1}{x^2-16} = -\infty$$

$$\lim_{x\to4^+} \frac{1}{x^2-16} = +\infty$$



(d) $y = f(x) = \dfrac{x^2}{x^2-16}$

Domain $x \neq \pm 4$

$$f'(x) = \frac{2x(x^2-16) - x^2 \cdot 2x}{(x^2-16)^2}$$

$$= \frac{-32x}{(x^2-16)^2}$$

$$f''(x) = \frac{-32(x^2-16)^2 + 32x \cdot 2(x^2-16) \cdot 2x}{(x^2-16)^4}$$

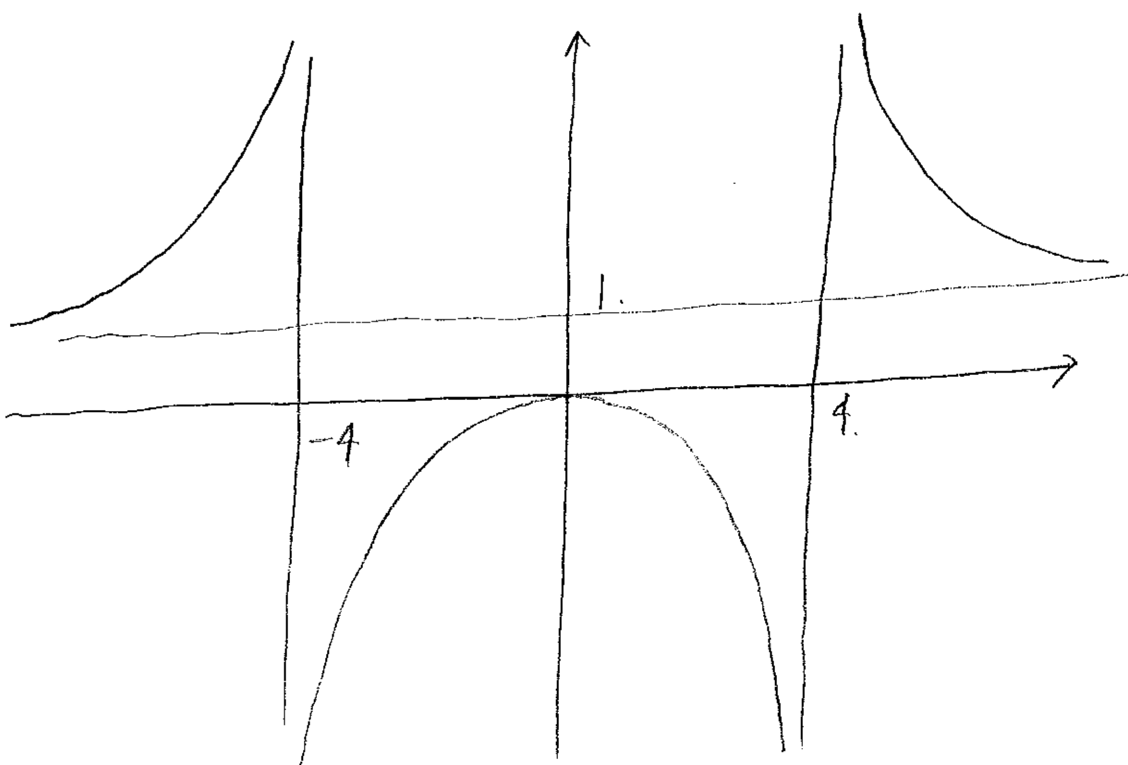
$$= \frac{-32\{(x^2-16) - 4x^2\}}{(x^2-16)^3}$$

$$= \frac{32(3x^2+16)}{(x^2-16)^3}$$

x		-4		0		4	
$f'(x)$	$+$	$\times$	$+$	0	$-$	$\times$	$-$
$f''(x)$	$+$	$\times$	$-$	$-$	$-$	$\times$	$+$
$f(x)$	$\nearrow$	$\times$	$\nearrow$	0	$\searrow$	$\times$	$\searrow$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-16} = 1$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2-16} = 1$$



(e) $\quad y = f(x) = e^{-x}\sin x$

on $[0, 2\pi]$

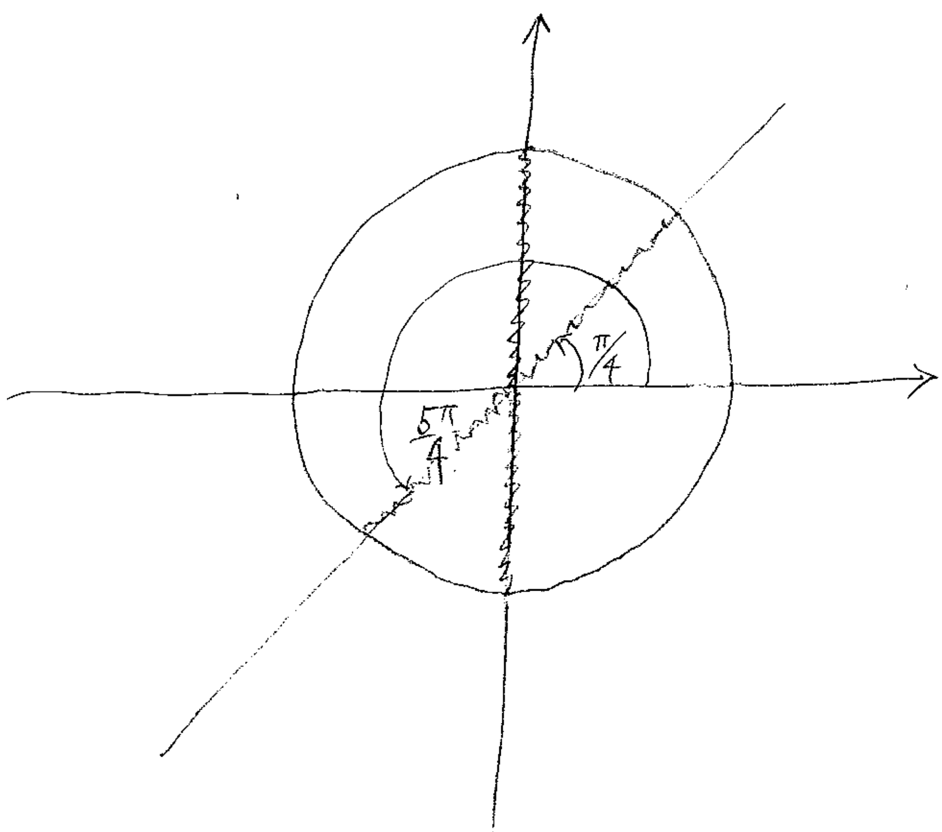
$$f'(x) = -e^{-x}\sin x + e^{-x}\cos x$$

$$= e^{-x}(\cos x - \sin x)$$

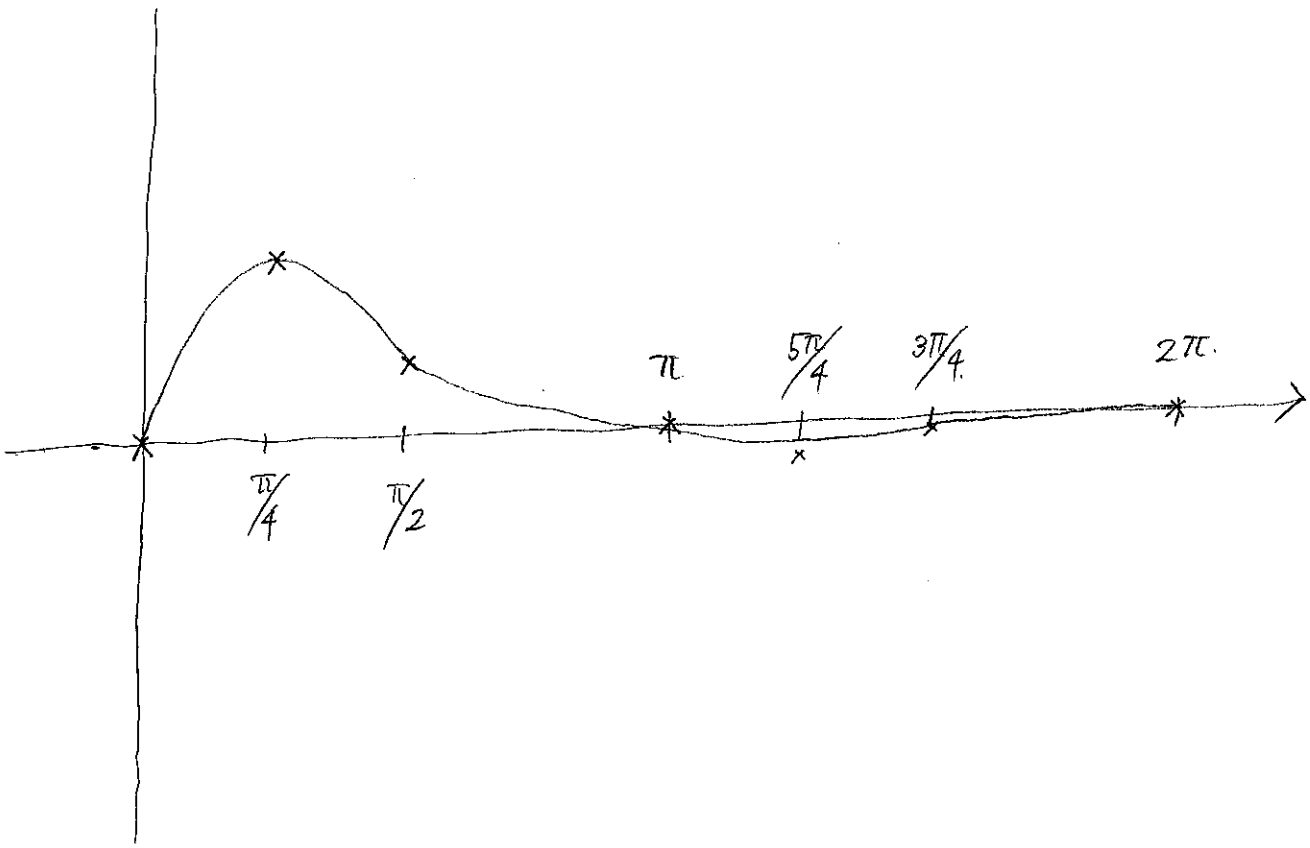
$$f''(x) = \cancel{e^{-x}\sin x} - e^{-x}\cos x$$

$$- e^{-x}\cos x \cancel{- e^{-x}\sin x}$$

$$= -2e^{-x}\cos x$$



x	0		$\pi/4$		$\pi/2$		$5\pi/4$		$3\pi/2$		2π
$f'(x)$		$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$	
$f''(x)$		$-$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$
$f(x)$	0	$\nearrow$	$e^{-\frac{\pi}{4}}\cdot\frac{\sqrt{2}}{2}$	$\searrow$	$e^{-\frac{\pi}{2}}$	$\searrow$	$-e^{-\frac{5\pi}{4}}\cdot\frac{\sqrt{2}}{2}$	$\nearrow$	$-e^{-\frac{3\pi}{2}}$	$\nearrow$	0



(f) $y = f(x) = \ln(x^2 - 10x + 24)$

Domain

$$x^2 - 10x + 24 > 0$$

$$\overset{\shortparallel}{(x-4)(x-6)}$$

$$\iff x < 4 \quad \text{or} \quad 6 < x$$

$$f'(x) = \frac{2x - 10}{x^2 - 10x + 24}$$

$$= \frac{2(x-5)}{x^2 - 10x + 24}$$

$$f''(x) = \frac{2\{1 \cdot (x^2 - 10x + 24) - (x-5)(2x-10)\}}{(x^2 - 10x + 24)^2}$$

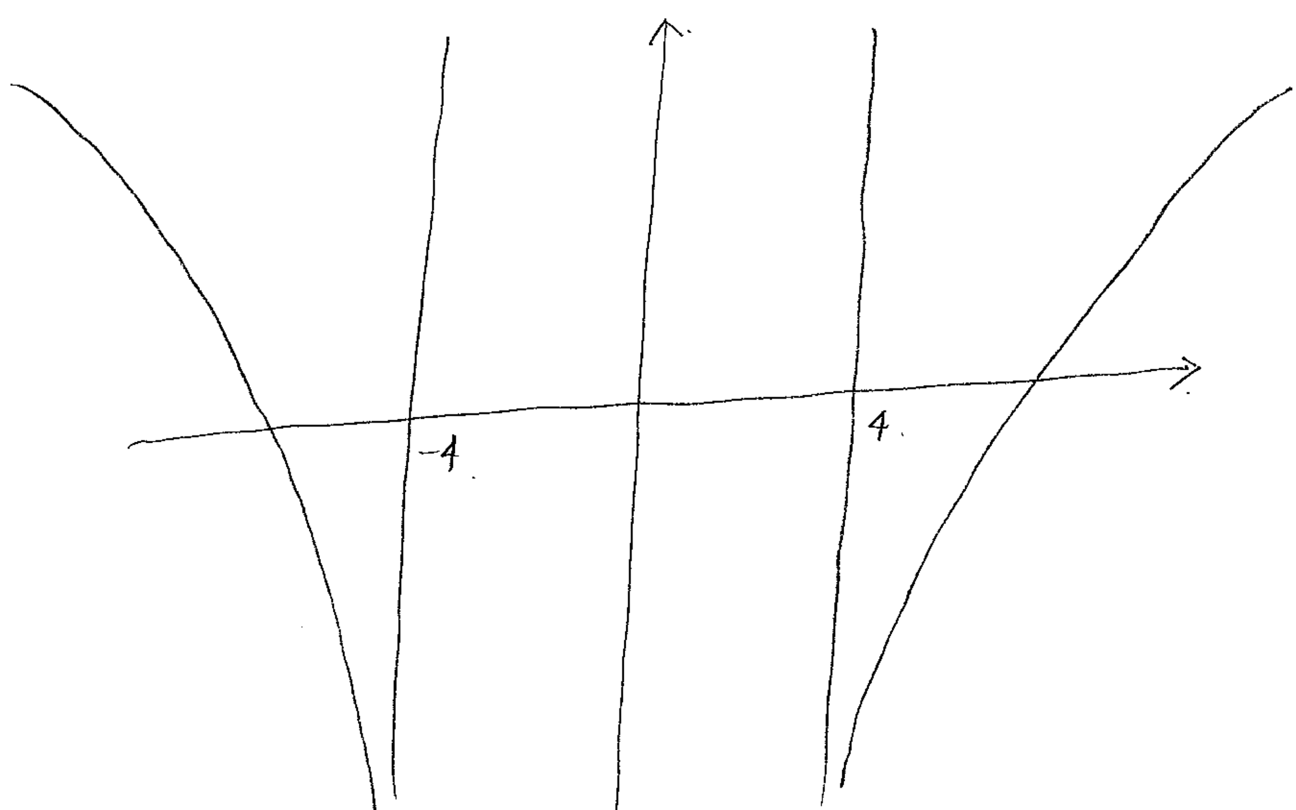
$$= \frac{-2(x^2 - 10x + 26)}{(x^2 - 10x + 24)^2}$$

$$= \frac{-2\{(x-5)^2 + 1\}}{(x^2 - 10x + 24)^2}$$

x		4		6	
$f'(x)$	$-$	$\times$	$\times$	$\times$	$+$
$f''(x)$	$-$	$\times$	$\times$	$\times$	$-$
$f(x)$	$\searrow$	$\times$	$\times$	$\times$	$\nearrow$

$$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \ln(x^2-10x+24) = -\infty$$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \ln(x^2-10x+24) = -\infty$$



9.1.

$$y = f(x) = \frac{2x^2}{x^2-1}.$$

$$= \frac{2x^2}{(x-1)(x+1)}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2}{x^2-1} = 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2}{x^2-1} = 2.$$

vertical asymptotes

$x = 1, \quad x = -1.$

horizontal asymptotes

$y = 2.$

9. 2.

$$y = f(x) = \ln\left(\frac{x^2+2x+3}{x^2-6x+8}\right)$$

$$x^2+2x+3 = (x+1)^2+2 > 2$$

$$x^2-6x+8 = (x-4)(x-2)$$

For $f(x)$ to be defined,

$$\frac{x^2+2x+3}{x^2-6x+8} > 0.$$

$\Leftrightarrow$

$$x < 2, \quad 4 < x$$

Domain

$$\lim_{x\to\infty} \ln\left(\frac{x^2+2x+3}{x^2-6x+8}\right) = 0$$

$$\lim_{x\to-\infty} \ln\left(\frac{x^2+2x+3}{x^2-6x+8}\right) = 0.$$

vertical asymptotes

$x = 2$, $x = 4$.

horizontal asymptote $y = 0$.

10.1.

$$f(x) = \frac{-3x^3 + 2x^2 + 7x - 5}{x^2 + x + 1}$$

$$\left(\begin{array}{r|l} & \quad -3x + 5 \\ \cline{2-2} x^2 + x + 1 & -3x^3 + 2x^2 + 7x - 5 \\ & -3x^3 - 3x^2 - 3x \\ \cline{2-2} & \qquad 5x^2 + 10x - 5 \\ & \qquad 5x^2 + 5x + 5 \\ \cline{2-2} & \qquad\qquad 5x - 10 \end{array} \right)$$

$$= -3x + 5 + \frac{5x - 10}{x^2 + x + 1}$$

$$\lim_{x \to \infty} \frac{5x - 10}{x^2 + x + 1} = 0$$

$$\lim_{x \to -\infty} \frac{5x - 10}{x^2 + x + 1} = 0$$

Slant asymptote

$$y = -3x + 5$$

10.2.

$$f(x) = \sqrt{x^2+2x-3}$$

$$= \sqrt{(x+1)^2-4}$$

$$= \sqrt{(x+1)^2 \left\{ 1 - \frac{4}{(x+1)^2} \right\}}$$

$(x \gg 0)$

$$= (x+1)\sqrt{1-\frac{4}{(x+1)^2}}$$

$(x \ll 0)$

$$= -(x+1)\sqrt{1-\frac{4}{(x+1)^2}}$$

Slant asymptotes

$$y = x+1.$$

&

$$y = -(x+1) = -x-1.$$

Check.

$$\lim_{x\to\infty}\left\{\sqrt{x^2+2x-3}-(x+1)\right\}=0$$

$$\lim_{x\to-\infty}\left\{\sqrt{x^2+2x-3}-(-x-1)\right\}=0$$

10.3

$$f(x) = \frac{2x^3 - 4x^2 + 5x - 10}{x^2 + x - 6}$$

$$= \frac{(x-2)(2x^2+5)}{(x-2)(x+3)}$$

$$
\begin{array}{r|l}
 & \quad 2x - 6 \\
\hline
x+3 & 2x^2 \qquad + 5 \\
 & 2x^2 + 6x \\
\hline
 & \quad -6x + 5 \\
 & \quad -6x - 18 \\
\hline
 & \qquad\quad 23
\end{array}
$$

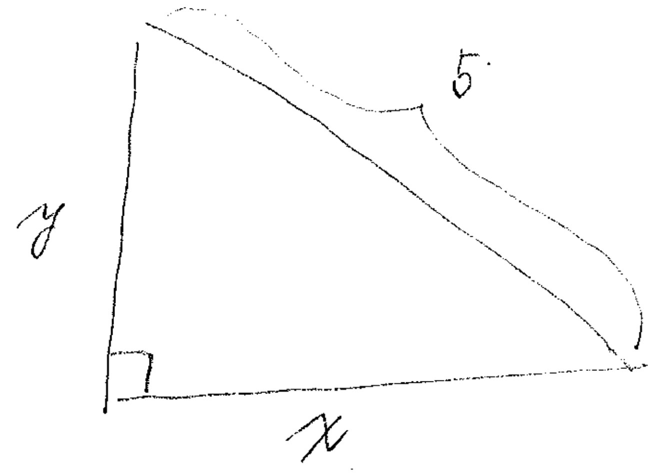
$$= 2x - 6 + \frac{23}{x+3}$$

vertical asymptote $x = -3$

horizontal asymptote NONE

slant asymptote $y = 2x - 6$

11.2 Picture



Condition $x^2 + y^2 = 5^2$

Maximize

$$A = \frac{1}{2} xy$$

Solution

$$A(x) = \frac{1}{2} x \sqrt{5^2 - x^2} \qquad 0 < x < 5$$

Observe maximizing $A(x)$ is equivalent to maximizing $G(x) = \{A(x)\}^2$

$$G(x) = \frac{1}{4} x^2 (5^2 - x^2) \qquad 0 < x < 5$$

$$= \frac{1}{4} (25 x^2 - x^4)$$

$$G'(x) = \frac{1}{4}\left(50x - 4x^3\right)$$

$$= -x\left(x^2 - \frac{25}{2}\right)$$

$$= -x\left(x + \frac{5}{\sqrt{2}}\right)\left(x - \frac{5}{\sqrt{2}}\right)$$

x	0		$\frac{5}{\sqrt{2}}$		5
$G'(x)$		$+$	0	$-$	
$G(x)$		$\nearrow$	max	$\searrow$	

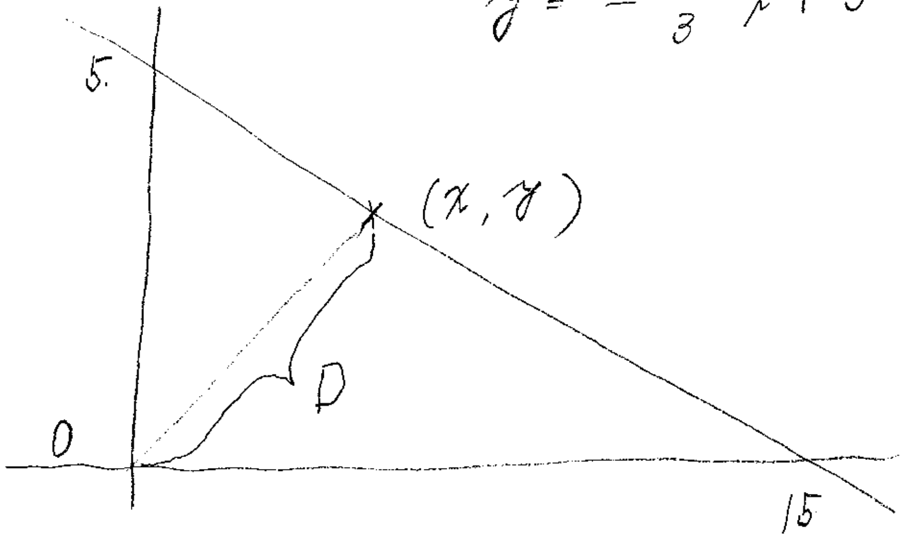
$A(x)$ is maximum when $x = \frac{5}{\sqrt{2}}$

$$A\left(\frac{5}{\sqrt{2}}\right) = \frac{1}{2}\,\frac{5}{\sqrt{2}}\sqrt{5^2 - \left(\frac{5}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{2}\,\frac{5}{\sqrt{2}}\cdot\frac{5}{\sqrt{2}} = \frac{25}{4}$$

11.3

$$y = -\frac{1}{3}x + 5$$



Minimize

$$D = \sqrt{x^2 + y^2}$$

Solution

$$D(x) = \sqrt{x^2 + \left(-\frac{1}{3}x + 5\right)^2}$$

Minimizing $D(x)$ is equivalent to minimizing $H(x) = \{D(x)\}^2$

$$H(x) = x^2 + \left(-\frac{1}{3}x + 5\right)^2 \qquad -\infty < x < \infty$$

$$= \frac{10}{9}x^2 - \frac{10}{3}x + 25$$

$$H'(x) = \frac{20}{9}x - \frac{10}{3}$$

$$= \frac{20}{9}\left(x - \frac{3}{2}\right)$$

x		$\frac{3}{2}$	
$H'(x)$	$-$	0	$+$
$H(x)$	$\searrow$		$\nearrow$

min.

The closest point is

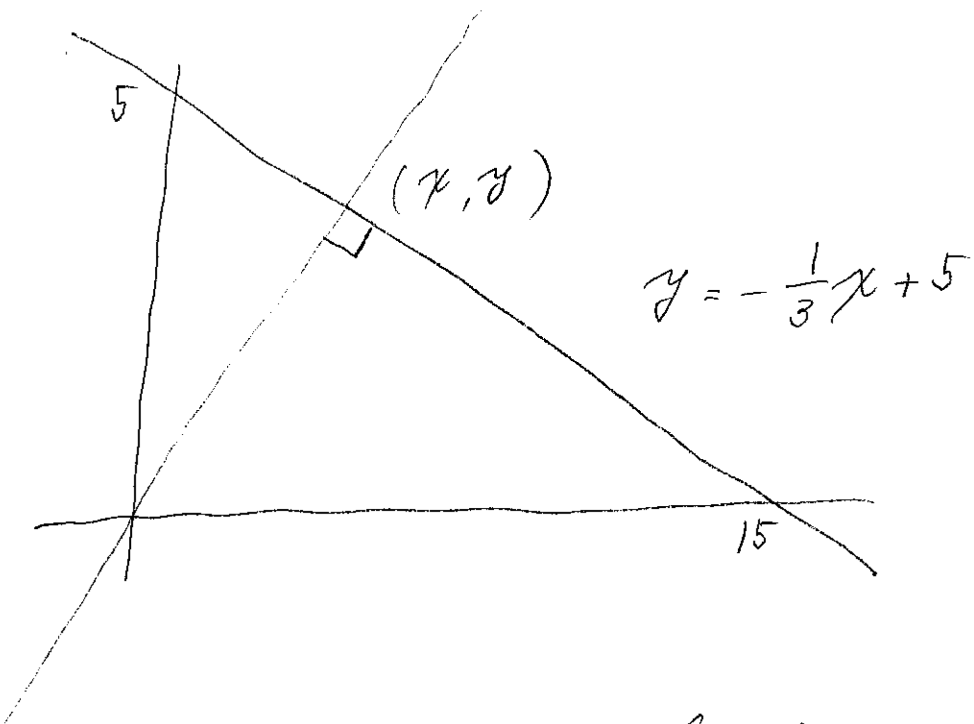
$$x = \frac{3}{2}$$

$$y = -\frac{1}{3}\left(\frac{3}{2}\right) + 5 = \frac{9}{2}$$

i.e.

$$\left(\frac{3}{2}, \frac{9}{2}\right)$$

Non calculus solution:



The closest point is the intersection

of $y = -\frac{1}{3}x + 5.$

& the line perpendicular to it

& passing through the origin

i.e

$y = 3x$

$-\frac{1}{3}x + 5 = 3x$

$5 = \frac{10}{3}x$

$x = \frac{3}{2}$

$y = -\frac{1}{3}\left(\frac{3}{2}\right) + 5 = \frac{9}{2}$

$\left(\frac{3}{2}, \frac{9}{2}\right)$