Study Guide for Exam 2

1. You are supposed to be able to use the chain rule properly and precisely, even when the function is obtained as the composition of several functions.

Example problems:

1.1. Compute the derivative of the following function:

(i) $y = \sin(\sin(\sin x))$

(ii) $y = \left(\dfrac{t-2}{2t+1}\right)^9$

(iii) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

(iv) $y = e^{\sec 3\theta}$

(v) $y = e^{2^{x^3}}$

(vi) $y = \sin^{-1}(\sqrt{1-x^2})$

(vii) $y = \sin^{-1}(\sqrt{1-x})$

(viii) $y = \ln|\sec(3\theta) + \tan(3\theta)|$

1.2. Suppose that $F(x) = f(x)^2 \cdot f(g(x))$ and that the functions f and g satisfy the following conditions. Find $F'(1)$.

$$\begin{cases} f(1) = 5, & f(2) = 3, & f(3) = -1 \\ f'(1) = 4, & f'(2) = 3, & f'(3) = -2 \\ g(1) = 3, & g(2) = 2, & g(3) = -1 \\ g'(1) = 2, & g'(2) = 3, & g'(3) = 4 \end{cases}$$

2. You are supposed to be able to determine the exact values of the formulas involving the trigonometric and inverse trigonometric functions. You should pay special attention to the range of the inverse trigonometric function. You need to know the double angle formulas for sine, cosine, and tangent.

Example Problems:

2.1. Find the exact values of the following expression.

(i) $\tan(\sin^{-1}\left(\dfrac{4}{5}\right)$

(ii) $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{3}\right)\right)$

(iii) $\tan^{-1}\left(\tan\left(\dfrac{5}{2}\right)\right)$

(iv) $\sin\left(2\sin^{-1}\left(\dfrac{12}{13}\right)\right)$

3. You are supposed to know how to compute the derivative of a function of the form $y = f(x)^{g(x)}$.

Example Problems:

3.1. Find the derivative of the following function.

(i) $y = x^x$

(ii) $y = (\ln x)^{\tan 3x}$

(iii) $y = (\sqrt{x})^{\sin x}$

(iv) $y = x^{1/x}$

4. You are supposed to understand the method of implicit differentiation to compute the derivative. For example, you should be able to determine the equation of the tangent line to the graph of a function implicitly defined, computing the derivative using the implicit differentiation.

Example Problems:

4.1. Suppose that f is a differentiable function defined on $(-\infty, \infty)$ satisfying the following equation

$$f(x) + x^2 (f(x))^3 = 10$$

and that $f(1) = 2$.

Find $f'(1)$.

4.2. Find the slope of the tangent to the curve given by the equation

$$x^2 + 2xy - y^2 + x = 2$$

at point $(x, y) = (1,\ 2)$

4.3. Find the equation of the tangent line to the curve $y^2(\ln x) + y = 3x$ at the point $(1, 3)$.

4.4. Find $\dfrac{dy}{dx}$ given $e^{x/y} = 7x - y$.

5. You are supposed to be able to provide an approximation of the value of a function, using the linear approximation.

Example Problems:

5.1. Find the linear approximation $L(x)$ of the function $f(x) = e^x$ at $a = 0$. Use this to estimate the value of $e^{0.01}$.

5.2. Estimate the value of $\sqrt[3]{26.8}$ using a linear approximation.

5.3 You borrowed \$100,000 from a bank at the annual compound rate of 3.5%. Compute how much you will owe to the bank after 30 years, using the linear approximation of the function $y = x^{30}$ at $a = 1$.

6. You are supposed to be able to determine when a particle is speeding up or down, whether it is accelerating or decelerating, given its position function. You are also supposed to be able to compute the total distance traveled during the given period.

Example Problems:

6.1. The position of a particle is given by the function

$$f(t) = t^3 - 15t^2 + 72t.$$

Determine the interval(s) ($0 \le t \le 8$) during which the particle is slowing down.

6.2. The position of a particle is given by the function

$$s = f(t) = t^3 - 6t^2 + 9t.$$

Find the total distance traveled during the first 6 seconds.

7. You are supposed to know how to provide the formulas for the trigonometric functions when the angles are given in the form of $\sin^{-1}(*), \cos^{-1}(*), \tan^{-1}(*)$.

Example Problems:

7.1. Find the formula for the following expression.

(i) $\tan\left(\sin^{-1} x\right)$

(ii) $\cos\left(\tan^{-1}\left(\frac{x}{\sqrt{9-x^2}}\right)\right)$

(iii) $\csc\left(\cot^{-1}\left(\frac{\sqrt{25-x^2}}{x}\right)\right)$ when $x > 0$.

Note: $\csc\left(\cot^{-1}\left(\frac{\sqrt{25-x^2}}{x}\right)\right)$ when $x < 0$, is a tricky problem. (Pay attention to the range of $\cot^{-1}$ in Page 66 of the textbook.)

8. You are supposed to know the definitions of the hyperbolic functions and be able to compute their derivatives, determine their exact values.

Example Problems:

8.1. Find the exact value f the following expression.

(i) $\sinh(0)$

(ii) $\sinh(\ln 5)$

(iii) $\cosh(\ln 5)$

(iv) $\dfrac{1+\tanh(\ln 2)}{1-\tanh(\ln 2)}$.

8.2. Find the derivative of the function $f(x) = \sinh(\ln x)$, and then evaluate $f(2)$ and $f'(2)$.

9. You are supposed to be able to compute the derivative of a function involving the logarithmic finctions, first simplifying the formula using the laws of the logarithms.

Example Problems:

9.1. Compute the derivatives of the following functions.

(i) $y = \ln(x\sqrt{x^2-10})$

(ii) $y = \ln(e^x + xe^x)$

10. FOUR "Related Rates" problems will be given in Exam 2. The problems are very similar to the ones given in the Webassign, and also to the examples discussed in the textbook. Of particular importance are:

- Snowball problem
- Light house problem
- Inverted circular conical tank problem (Gravel pile problem)
- Kite problem
- Ladder problem
- Man walking away from the light problem
- Two cars (ships) moving east-west, south-north problem
- Trough problem
- Particle moving along the graph of a function problem

10.1. If a snowball melts so that its surface area decreases at a rate of 3 cm^2/min, find the rate at which the diameter decreases when the diameter is 11 cm.

10.2. A light house is located on a small island 4 km away from the nearest point P on a straight shoreline and its light makes 5 revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P ?

10.3. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

10.4. A kite 100 ft above the ground moves horizontally at a speed of 3 ft/sec. At what rate is the angle (in radians) between the string and the horizontal decreasing when 200 ft of string have been let out ?

10.5. (speed) A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/sec, how fast is the top pf the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall ?

10.5 (angle). A 15-foot plank of wood is leaning against a vertical wall and its bottom is being pushed toward the wall at the rate of 2 ft/sec.

At what rate is the angle θ between the plank and the ground changing when the acute angle the plank makes with the ground is $\pi/4$?

10.6. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the street light. If the length of the man's shadow is increasing at the rate of 2 ft/sec, how fast is he walking?

Warning: This problem is DIFFERENT from #1 in the Webassign HW 16.

10.7. At noon, ship A is 180 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 30 km/h. How fast is the distance between the ships changing at 4:00 PM.

10.8. A trough is 8 ft long and its ends have the shape of isosceles triangles that are 5 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 11 ft^3/min, how fast is the water level rising when the water is 4 inches deep ?

10.9. A particle is moving along the curve $xy = 12$. As it reaches the point $(6, 2)$, the x-coordinate is decreasing at a rate of 5 cm/sec. What is the rate of change of the y-coordinate of the particle at that instant ?

Note: The unit for measuring the coordinate length is given by "cm".